Complete Solutions to Exercises 2.4

1. (a) We are asked to find $\left[45,\quad 81\right]$. The prime factors of 45 and 81 are

$$45 = 3^2 \times 5 \text{ and } 81 = 3^4$$

Applying Proposition (2.19):

$$\left[a,\quad b\right] = p_1^{\max\left(e_1,\, f_1\right)} \times p_2^{\max\left(e_2,\, f_2\right)} \times p_3^{\max\left(e_3,\, f_3\right)} \times \cdots \times p_k^{\max\left(e_k,\, f_k\right)}$$

Gives

$$\begin{aligned}\left[45,\quad 81\right] &= \left[3^2 \times 5,\quad 3^4\right] \\ &= 3^{\max\left(2,\,4\right)} \times 5^{\max\left(1,\,0\right)} = 3^4 \times 5^1 = 405\end{aligned}$$

(b) Similarly for $\left[2000,\quad 2015\right]$ we have

$$2000 = 2 \times 1000 = 2 \times 10^3 = 2 \times \left(2 \times 5\right)^3 = 2^4 \times 5^3$$

$$2015 = 5 \times 403 = 5 \times 13 \times 31$$

Using the above proposition we have

$$\begin{aligned}\left[2000,\quad 2015\right] &= \left[2^4 \times 5^3,\quad 5 \times 13 \times 31\right] \\ &= 2^{\max\left(4,\,0\right)} \times 5^{\max\left(3,\,1\right)} \times 13^{\max\left(0,\,1\right)} \times 31^{\max\left(0,\,1\right)} \\ &= 2^4 \times 5^3 \times 13^1 \times 31^1 = 806\ 000\end{aligned}$$

We have $\left[2000,\quad 2015\right] = 806\ 000$.

(c) *What do you notice about the two given integers* $\left[1000,\quad 1001\right]$ *?*

1000 and 1001 are *relatively prime* which means they have *no* factor $\left(> 1\right)$ in common. Using Proposition (2.20):

Let a and b be relatively prime integers then $\left[a,\quad b\right] = a \times b$.

To the given integers yields

$$\left[1000,\quad 1001\right] = 1000 \times 1001 = 1\ 001\ 000$$

2. We need to find the LCM of 10 and 8. In this case it is easier to make a list of the multiples of 10 and 8:

$$10, 20, 30, 40, 50, \cdots \text{ and } 8, 16, 24, 32, 40, 48, \cdots$$

Hence $\left[10,\quad 8\right] = 40$ so we need to purchase 4 packages of hotdogs and 5 packages of buns.

3. We need to find the LCM of 85 and 91. The prime decompositions of these numbers are

$$85 = 5\times 17 \text{ and } 91 = 7\times 13$$

Hence 85 and 91 are relatively prime because they have no factor greater than 1 in common:

$$\left[85,\quad 91\right] = 85\times 91 = 7735$$

We need to compare the fractions $\frac{64}{85}$ and $\frac{69}{91}$:

$$\frac{64}{85} = \frac{64\times 91}{7735} = \frac{5824}{7735} \qquad \text{[Mechanics]}$$

$$\frac{69}{91} = \frac{69\times 85}{7735} = \frac{5865}{7735} \qquad \text{[Real Analysis]}$$

Therefore Harry performed better on the real analysis paper.

4. (i) In order to find $\left[20,\quad 265,\quad 530\right]$ we use Proposition (2.23):

$$\boxed{\left[a_1,\ a_2,\ a_3,\cdots,a_n\right] = \left[\left[a_1,\ a_2,\ a_3,\cdots,a_{n-1}\right],\quad a_n\right]}$$

First we find $\left[20,\quad 265\right]$. The prime decomposition of these integers is given by

$$20 = 2^2\times 5 \text{ and } 265 = 5\times 53$$

By applying Proposition (2.19):

$$\boxed{\left[a,\quad b\right] = p_1^{\max(e_1,\, f_1)}\times p_2^{\max(e_2,\, f_2)}\times p_3^{\max(e_3,\, f_3)}\times\cdots\times p_k^{\max(e_k,\, f_k)}}$$

We have

$$\begin{aligned}\left[20,\quad 265\right] = \left[2^2\times 5,\quad 5\times 53\right] &= 2^{\max(2,\,0)}\times 5^{\max(1,\,1)}\times 53^{\max(0,\,1)}\\ &= 2^2\times 5\times 53 = 2\times 10\times 53 = 1060\end{aligned}$$

We using the above Proposition (2.23) we have

$$\begin{aligned}\left[20,\quad 265,\quad 530\right] &= \left[\left[20,\quad 265\right],\quad 530\right]\\ &= \left[1060,\quad 530\right] = 1060 \qquad \left[\text{because } 1060 = 2\times 530\right]\end{aligned}$$

Hence $\left[20,\quad 265,\quad 530\right] = 1060$.

You could also evaluate this directly as follows:

$$\begin{aligned}\left[20,\quad 265,\quad 530\right] &= \left[2^2\times 5,\quad 5\times 53,\quad 2\times 5\times 53\right]\\ &= 2^{\max(2,\ 0,\ 1)}\times 5^{\max(1,\ 1,\ 1)}\times 53^{\max(0,\ 1,\ 1)}\\ &= 2^2\times 5\times 53 = 1060\end{aligned}$$

Hence [20, 265, 530] = 1060.

(ii) We are asked to find $\dfrac{1}{20}+\dfrac{1}{265}+\dfrac{1}{530}$. We use the result of part (i):

$\left[20,\ \ 265,\ \ 530\right]=1060$ to convert these fractions into a common denominator

$$\frac{1}{20}=\frac{1\times 53}{20\times 53}=\frac{53}{1060}$$
$$\frac{1}{265}=\frac{4\times 1}{4\times 265}=\frac{4}{1060}$$
$$\frac{1}{530}=\frac{2\times 1}{2\times 530}=\frac{2}{1060}$$

Adding these gives

$$\frac{1}{20}+\frac{1}{265}+\frac{1}{530}=\frac{53+4+2}{1060}=\frac{59}{1060}$$

5. (i) Since the integers 3 and 4 are relatively prime so

$$\left[3,\ \ 4\right]=3\times 4=12$$

Using Proposition (2.23):

$$\boxed{\left[a_1,\ \ a_2,\ \ a_3,\cdots,\ a_n\right]=\left[\left[a_1,\ \ a_2,\ \ a_3,\cdots,\ a_{n-1}\right],\ \ a_n\right]}$$

We have

$$\left[3,\ \ 4,\ \ 28\right]=\left[12,\ \ 28\right]$$

The multiples of 28 are 28, 56, 84 and 84 is also a multiple of 12 so

$$\left[3,\ \ 4,\ \ 28\right]=\left[12,\ \ 28\right]=84$$

(ii) We need to solve $\dfrac{1}{3}+\dfrac{1}{4}+\dfrac{1}{28}+x=1$. Transposing gives

$$\begin{aligned}x&=1-\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{28}\right)\\&=1-\left(\frac{28}{84}+\frac{21}{84}+\frac{3}{84}\right)\\&=1-\left(\frac{28+21+3}{84}\right)=1-\frac{52}{84}=\frac{32}{84}=\frac{8}{21}\end{aligned}$$

6. (a) We need to find $\left[60,\ \ 100\right]$. The prime decompositions of these integers are

$$60=2^2\times 3\times 5\ \text{ and }\ 100=10^2=2^2\times 5^2.$$

Applying Proposition (2.19):

$$\boxed{\left[a,\ \ b\right]=p_1^{\max\left(e_1,\,f_1\right)}\times p_2^{\max\left(e_2,\,f_2\right)}\times p_3^{\max\left(e_3,\,f_3\right)}\times\cdots\times p_k^{\max\left(e_k,\,f_k\right)}}$$

We have

$$\begin{aligned}\left[2^2\times 3\times 5,\quad 2^2\times 5^2\right] &= 2^{\max(2,\,2)}\times 3^{\max(1,\,0)}\times 5^{\max(1,\,2)}\\ &= 2^2\times 3\times 5^2 = 300\end{aligned}$$

Hence $\left[60,\quad 100\right]=300$.

(b) We need to find $\left[600,\quad 1000\right]$. Similarly we have

$$600 = 2^3\times 3\times 5^2 \text{ and } 1\,000 = 2^3\times 5^3$$

Using Proposition (2.19) we have

$$\begin{aligned}\left[2^3\times 3\times 5^2,\quad 2^3\times 5^3\right] &= 2^{\max(3,\,3)}\times 3^{\max(1,\,0)}\times 5^{\max(2,\,3)}\\ &= 2^3\times 3\times 5^3 = 3\,000\end{aligned}$$

Therefore $\left[600,\quad 1\,000\right]=3\,000$.

(c) We must determine $\left[6\,000,\quad 10\,000\right]$. We have

$$6000 = 2^4\times 3\times 5^3 \text{ and } 10\,000 = 2^4\times 5^4$$

Using Proposition (2.19) we have

$$\begin{aligned}\left[2^4\times 3\times 5^3,\quad 2^4\times 5^4\right] &= 2^{\max(4,\,4)}\times 3^{\max(1,\,0)}\times 5^{\max(3,\,4)}\\ &= 2^4\times 3\times 5^4 = 30\,000\end{aligned}$$

Therefore $\left[6\,000,\quad 10\,000\right]=30\,000$.

If the pair of integers are 10 times larger than the corresponding LCM is also 10 times larger.

7. We need to show that $\left[ab,\quad ac\right]=a\times\left[b,\quad c\right]$ given that a, b and c are positive integers.

Proof.

By Proposition (2.22):

$$\boxed{\left[x,\quad y\right]=\frac{x\times y}{\gcd\left(x,\quad y\right)}}$$

And using the given hint $\gcd\left(dx,\quad dy\right)=\left|d\right|\gcd\left(x,\quad y\right)$ where $d\neq 0$ we have

$$\left[ab,\quad ac\right] = \frac{ab \times ac}{\gcd\left(ab,\quad ac\right)}$$

$$= \frac{ab \times ac}{\left|a\right|\gcd\left(b,\quad c\right)} \qquad \left[\text{By hint}\right]$$

$$\underbrace{=}_{\substack{\text{Because } a \text{ is positive}\\ \text{so } |a|=a}} \frac{\cancel{a}b \times ac}{\cancel{a}\gcd\left(b,\quad c\right)} = a \times \frac{b \times c}{\gcd\left(b,\quad c\right)}$$

Applying the above Proposition (2.22) again:

$$\left[b,\quad c\right] = \frac{b \times c}{\gcd\left(b,\quad c\right)}$$

Substituting this $\left[b,\quad c\right] = \dfrac{b \times c}{\gcd\left(b,\quad c\right)}$ into $\left[ab,\quad ac\right] = a \times \dfrac{b \times c}{\gcd\left(b,\quad c\right)}$ gives

$$\left[ab,\quad ac\right] = a \times \frac{b \times c}{\gcd\left(b,\quad c\right)} = a \times \left[b,\quad c\right]$$

This is our required result so it completes our proof.

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8. We are asked to prove $\left[p,\quad q\right] = p \times q$ where $p \neq q$ and are primes.

Proof.

By result of question 4 of Exercises 2a we have

p and q be distinct primes then $\gcd\left(p,\quad q\right) = 1$

Since $\gcd\left(p,\quad q\right) = 1$ so applying Proposition (2.20):

Let a and b be relatively prime positive integers then $\left[a,\quad b\right] = a \times b$.

To $\left[p,\quad q\right]$ gives $\left[p,\quad q\right] = p \times q$.

This completes our proof.

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9. We need to prove $\left[a,\quad ma\right] = ma$.

Proof.

First note that $\gcd\left(a,\quad ma\right) = a$. Using Proposition (2.22):

$$\boxed{\left[x,\quad y\right] = \frac{x \times y}{\gcd\left(x,\quad y\right)}}$$

We have

$$\left[a,\quad ma\right]=\frac{a\times ma}{\gcd\left(a,\quad ma\right)}=\frac{ma^{\cancel{2}}}{\cancel{a}}=ma\qquad\text{[Cancelling]}$$

This completes our proof.

■

10. We are asked to prove $\left[a,\quad bc\right]=a\times b\times c$ given that a, b and a, c are relatively prime.

Proof.

Using the given hint we have

$$\gcd\left(a,\quad bc\right)=1$$

Applying Proposition (2.20):

Let x and y be relatively prime positive integers then $\left[x,\quad y\right]=x\times y$.

To $\left[a,\quad bc\right]$ gives $\left[a,\quad bc\right]=a\times b\times c$. This completes our proof.

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11. To disprove something we need to produce a counter example.

(a) To disprove $\left[p,\quad p\right]=p^2$ we let $p=3$ then

$$\left[3,\quad 3\right]=3 \text{ not } 3^2.$$

(b) We are asked to disprove $\left[a,\quad b\right]=a\times b$. Let $a=6$ and $b=9$ then

$$\left[6,\quad 9\right]=18\neq 6\times 9\qquad\left[\text{Not Equal}\right]$$

(c) We need to disprove the following statement;

If $\left[a,\quad b\right]=n$ and $\left[b,\quad c\right]=m$ then $\left[a,\quad c\right]=m\times n$.

Let $a=6,\ b=8$ and $c=9$ then

$$\left[6,\quad 8\right]=24 \text{ and } \left[8,\quad 9\right]=72$$

However $\left[6,\quad 9\right]=18$ and $18\neq 24\times 72$ [Not Equal].

(d) We have to disprove $\left[a+b,\quad c\right]=\left[a,\quad c\right]+\left[b,\quad c\right]$.

Let $a=6,\ b=8$ and $c=9$ then

$$\left[6+8,\quad 9\right]=\left[14,\quad 9\right]=126$$
$$\left[6,\quad 9\right]+\left[8,\quad 9\right]=18+72=90$$

Since $90\neq 126$ so the following statement

$$\left[a+b,\quad c\right]=\left[a,\quad c\right]+\left[b,\quad c\right] \text{ is } \textit{false}$$

(e) We are asked to disprove $\left[ab,\quad ac\right]=a^2\left[b,\quad c\right]$.

Let $a=6,\ b=8$ and $c=9$ then

$$\begin{aligned}\left[ab,\quad ac\right]&=\left[6\times 8,\quad 6\times 9\right]=\left[48,\quad 54\right]=432\\ a^2\times\left[b,\quad c\right]&=6^2\times\left[8,\quad 9\right]=36\times 72=2\ 592\end{aligned}$$

Actually by the result of question 7 we have $\left[ab,\quad ac\right]=a\left[b,\quad c\right]$ provided a is positive.

(f) We need to disprove $\gcd\left(a,\quad b,\quad c\right)\times\left[a,\quad b,\quad c\right]=a\times b\times c$.

Let $a=6,\ b=8$ and $c=9$ then

$$\gcd\left(6,\quad 8,\quad 9\right)=1 \text{ and } \left[6,\quad 8,\quad 9\right]=72$$

We have

$$\gcd\left(6,\quad 8,\quad 9\right)\times\left[6,\quad 8,\quad 9\right]=1\times 72=72$$

But $6\times 8\times 9=432$. Hence

$$\gcd\left(a,\quad b,\quad c\right)\times\left[a,\quad b,\quad c\right]\neq a\times b\times c \qquad \left[\text{Not Equal}\right]$$

Note that this result holds for two positive integers; $\gcd\left(a,\quad b\right)\times\left[a,\quad b\right]=a\times b$ but is false for three positive integers a, b and c. (See result of question 22.)

12. Required to prove that the LCM of two positive integers is unique.

Proof.

Let a, b be positive integers whose LCM is given by

$$\left[a,\quad b\right]=c$$

Suppose $\left[a,\quad b\right]=d$ where $d\neq c$.

If $d>c$ then $\left[a,\quad b\right]$ *cannot* equal d. *Why not?*

Because by Definition (2.18) part (ii):

> Let $\left[a,\quad b\right]=m$. Then m satisfies
>
> (ii) if both $a\mid n$ and $b\mid n$ then $m\leq n$ - least multiple.

The smallest multiple is c in this case as $c<d$.

If $d<c$ then $\left[a,\quad b\right]$ *cannot* equal c because of the above definition, we have a smaller common multiple d.

In either case where $d > c$ or $d < c$ we have a contradiction, so $d = c$ which implies that $\left[a,\ \ b\right]$ is unique.

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13. For this question we could use Proposition (2.23):

$$\left[a_1,\ a_2,\ a_3,\cdots,a_n\right]=\left[\left[a_1,\ a_2,\ a_3,\cdots,a_{n-1}\right],\ \ a_n\right]$$

Or the prime decomposition which is generally easier for our given smaller numbers.

(a) We need to find $\left[2,\ \ 3,\ \ 5,\ \ 7\right]$. In this case as all the numbers are distinct primes so they are relatively prime to each other (pairwise prime) which means we can use the following:

$$\left[2,\ \ 3,\ \ 5,\ \ 7\right]=2\times 3\times 5\times 7=210$$

(b) This time we use the prime decomposition method:

We need to find $\left[24,\ \ 35,\ \ 51,\ \ 64\right]$. Writing the prime decompositions of each number gives

$$24 = 2^3\times 3\,,\ 35=5\times 7\,,\ 51=3\times 17 \text{ and } 64 = 2^6$$

We have

$$\begin{aligned}\left[24,\ \ 35,\ \ 51,\ \ 64\right]&=\left[2^3\times 3,\ \ 5\times 7,\ \ 3\times 17,\ \ 2^6\right]\\&=2^{\max(3,\ 0,\ 0,\ 6)}\times 3^{\max(1,\ 0,\ 1,\ 0)}\times 5^{\max(0,\ 1,\ 0,\ 0)}\times 7^{\max(0,\ 1,\ 0\ \ 0)}\times 17^{\max(0,\ 0,\ 1,\ 0)}\\&=2^6\times 3^1\times 5^1\times 7^1\times 17^1=114\ 240\end{aligned}$$

Hence $\left[24,\ \ 35,\ \ 51,\ \ 64\right]=114\ 240$.

(c) We are asked to find $\left[11,\ \ 121,\ \ 132,\ \ 99,\ \ 77\right]$.

Writing the prime decompositions of each of these numbers;

$$11=11,\ 121=11^2,\ 132=12\times 11=2^2\times 3\times 11\,,\ \ 99=3^2\times 11 \text{ and } 77=7\times 11$$

Therefore we have

$$\begin{aligned}\left[11,\ \ 121,\ \ 132,\ \ 99,\ \ 77\right]&=\left[11,\ \ 11^2,\ \ 2^2\times 3\times 11,\ \ 3^2\times 11,\ \ 7\times 11\right]\\&=2^2\times 3^2\times 7\times 11^2=30\ 492\end{aligned}$$

Hence $\left[11,\ \ 121,\ \ 132,\ \ 99,\ \ 77\right]=30\ 492$.

14. In this case we need to find the LCM of 6, 8 and 11:

$$\begin{aligned}\left[6,\ \ 8,\ \ 11\right]&=\left[\left[6,\ \ 8\right],\ \ 11\right]\\&=\left[24,\ \ 11\right]=24\times 11=264\end{aligned}$$

Since we are given that the number of soldiers in the battalion is between 500 and 600 and the remainder is 3 so

$$\text{Number of soldiers } = (2 \times 264) + 3 = 531.$$

15. We are asked to prove the following:

> Let $a_1,\ a_2,\ a_3, \cdots, a_n$ be *pairwise relatively prime* integers then
> $$[a_1,\ a_2,\ a_3, \cdots, a_n] = a_1 \times a_2 \times \cdots \times a_n$$

How do we prove this?

By mathematical induction.

Proof.

Base case $n = 2$:

By Proposition (2.20):

> Let a and b be relatively prime integers then $[a,\ b] = a \times b$.

We have our result for $n = 2$; that is

$$[a_1,\ a_2] = a_1 \times a_2$$

Assume the result is true for $n = k$:

$$[a_1,\ a_2,\ a_3, \cdots, a_k] = a_1 \times a_2 \times \cdots \times a_k \qquad (*)$$

Required to prove that

$$[a_1,\ a_2,\ a_3, \cdots, a_k,\ a_{k+1}] = a_1 \times a_2 \times \cdots \times a_k \times a_{k+1}$$

By applying Proposition (2.23):

> $$[a_1,\ a_2,\ a_3, \cdots, a_n] = [[a_1,\ a_2,\ a_3, \cdots, a_{n-1}],\ a_n]$$

To the above $[a_1,\ a_2,\ a_3, \cdots, a_k,\ a_{k+1}]$ gives

$$\begin{aligned} [a_1,\ a_2,\ a_3, \cdots, a_k,\ a_{k+1}] &= [[a_1,\ a_2,\ a_3, \cdots, a_k],\ a_{k+1}] \\ &= [a_1 \times a_2 \times \cdots \times a_k,\ a_{k+1}] \qquad [\text{By } (*)] \end{aligned}$$

We are given that the a integers are *pairwise relatively prime* which implies we have

$$\gcd(a_1,\ a_{k+1}) = \gcd(a_2,\ a_{k+1}) = \cdots = \gcd(a_k,\ a_{k+1}) = 1$$

Using the given hint:

> If $\gcd(a_1,\ b) = \cdots = \gcd(a_n,\ b) = 1$ then $\gcd(a_1 \times a_2 \times \cdots \times a_n,\ b) = 1$.

And by the above Proposition (2.20) on the above derivation gives

$$\left[a_1,\ a_2,\ a_3, \cdots, a_k,\ a_{k+1}\right] = \left[a_1 \times a_2 \times \cdots \times a_k,\ a_{k+1}\right]$$
$$= a_1 \times a_2 \times \cdots \times a_k \times a_{k+1}$$

By mathematical induction we have our required result.

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16. We are asked to prove the following:

> Let $a = p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \times \cdots \times p_k^{e_k}$ and $b = p_1^{f_1} \times p_2^{f_2} \times p_3^{f_3} \times \cdots \times p_k^{f_k}$ be the prime decompositions of a and b and $e_j \geq 0$ and $f_j \geq 0$. Then the LCM is given by
> $$\left[a,\ b\right] = p_1^{\max(e_1,\, f_1)} \times p_2^{\max(e_2,\, f_2)} \times p_3^{\max(e_3,\, f_3)} \times \cdots \times p_k^{\max(e_k,\, f_k)}$$

Proof.

Let the prime decompositions of a and b be given by

$$a = p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \times \cdots \times p_k^{e_k} \text{ and } b = p_1^{f_1} \times p_2^{f_2} \times p_3^{f_3} \times \cdots \times p_k^{f_k}$$

Let $\left[a,\ b\right] = m$. Required to prove that

$$m = p_1^{\max(e_1,\, f_1)} \times p_2^{\max(e_2,\, f_2)} \times p_3^{\max(e_3,\, f_3)} \times \cdots \times p_k^{\max(e_k,\, f_k)}$$

Since m is a multiple of both integers a and b so it must have *all* these primes and *no* others (least multiple):

$$m = p_1^{j_1} \times p_2^{j_2} \times p_3^{j_3} \times \cdots \times p_k^{j_k}$$

Why?

Because if a prime p_n is missing from m then m *cannot* be a multiple of both given integers a and b. This implies that m *cannot* be the LCM of a and b.

To complete the proof we need to show that the indices

$$j_1 = \max\left(e_1,\, f_1\right),\ \ j_2 = \max\left(e_2,\, f_2\right),\ \ \cdots,\ \ j_k = \max\left(e_k,\, f_k\right)$$

Let us consider the first index, $\max\left(e_1,\, f_1\right)$.

We consider two cases; $j_1 > \max\left(e_1,\, f_1\right)$ and then $j_1 < \max\left(e_1,\, f_1\right)$. In each case we derive a contradiction.

<u>Case 1</u>:

If $j_1 > \max\left(e_1,\, f_1\right)$ (j_1 is greater than e_1 or f_1) then let

$$n = p_1^{j_1} \times p_2^{\max(e_2,\, f_2)} \times p_3^{\max(e_3,\, f_3)} \times \cdots \times p_k^{\max(e_k,\, f_k)}$$

Our a is given by the prime decomposition:

$$a = p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \times \cdots \times p_k^{e_k} \text{ which implies } a \,\Big|\, n$$

Similarly, we have $b \,\Big|\, n$. Therefore n is a common multiple of a and b.

Also

$$\begin{aligned} n &= p_1^{j_1} \times p_2^{\max(e_2,\, f_2)} \times p_3^{\max(e_3,\, f_3)} \times \cdots \times p_k^{\max(e_k,\, f_k)} \\ &= p_1^{j_1 - \max(e_1,\ f_1) + \max(e_1,\ f_1)} \times p_2^{\max(e_2,\, f_2)} \times p_3^{\max(e_3,\, f_3)} \times \cdots \times p_k^{\max(e_k,\, f_k)} \\ &= p_1^{j_1 - \max(e_1,\ f_1)} \times \left[p_1^{\max(e_1,\ f_1)} \times p_2^{\max(e_2,\, f_2)} \times p_3^{\max(e_3,\, f_3)} \times \cdots \times p_k^{\max(e_k,\, f_k)} \right] \qquad (*) \end{aligned}$$

Let

$$m' = p_1^{\max(e_1,\, f_1)} \times p_2^{\max(e_2,\, f_2)} \times p_3^{\max(e_3,\, f_3)} \times \cdots \times p_k^{\max(e_k,\, f_k)}$$

Then m' is also a common multiple of a and b because

$$a = p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \times \cdots \times p_k^{e_k} \text{ and } b = p_1^{f_1} \times p_2^{f_2} \times p_3^{f_3} \times \cdots \times p_k^{f_k}$$

Substituting this into (*) gives

$$n = p_1^{j_1 - \max(e_1,\ f_1)} \times m' \text{ which implies } n > m'$$

By the definition of the LCM we conclude that n *cannot* be the *least common multiple of* a and b. *Why not?*

Because the *least* common multiple $n >$ common multiple m'.

Hence with $j_1 > \max(e_1,\, f_1)$ we have $[a,\ b] \neq n$ where

$$n = p_1^{j_1} \times p_2^{\max(e_2,\, f_2)} \times p_3^{\max(e_3,\, f_3)} \times \cdots \times p_k^{\max(e_k,\, f_k)}$$

<u>Case 2</u>:

If $j_1 < \max(e_1,\, f_1)$ (j_1 is less than e_1 or f_1) and without loss of generality assume $\max(e_1,\, f_1) = e_1$. From this we have $j_1 < \max(e_1,\, f_1) = e_1$ or $j_1 < e_1$. Let

$$n = p_1^{j_1} \times p_2^{\max(e_2,\, f_2)} \times p_3^{\max(e_3,\, f_3)} \times \cdots \times p_k^{\max(e_k,\, f_k)}.$$

Then n is *not* a multiple of a. *Why not?*

Suppose n is a multiple of a or a is divisor of n, that is $a \mid n$.

We are given that $a = p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \times \cdots \times p_k^{e_k}$ so $p_1^{e_1} \mid a$ and

$$a \mid n \text{ implies } p_1^{e_1} \,\Big|\, \left(p_1^{j_1} \times p_2^{\max(e_2,\, f_2)} \times \cdots \times p_k^{\max(e_k,\, f_k)} \right).$$

Remember these are *all* distinct primes so

$$\gcd\left(p_1^{e_1},\ p_2^{\max(e_2,\, f_2)}\right) = \cdots = \gcd\left(p_1^{e_1},\ p_k^{\max(e_k,\, f_k)}\right) = 1$$

Then by the result of question 24 of Supplementary Problems 1:

If $a \mid (b_1 \times b_2 \times \cdots \times b_n)$ and $\gcd(a,\ b_1) = \gcd(a,\ b_2) = \cdots = \gcd(a,\ b_{n-1}) = 1$ then $a \mid b_n$.

Applying this to

$$p_1^{e_1} \,\Big|\, \left(p_1^{j_1} \times p_2^{\max(e_2,\, f_2)} \times \cdots \times p_k^{\max(e_k,\, f_k)}\right) \text{ gives } p_1^{e_1} \,\Big|\, p_1^{j_1}$$

This $p_1^{e_1} \mid p_1^{j_1}$ *cannot* be right because we are supposing $j_1 < e_1$ or $e_1 > j_1$.

Hence our supposition $j_1 < \max(e_1, f_1)$ must be wrong.

Combining both cases together we have

$$j_1 \not> \max(e_1, f_1) \text{ and } j_1 \not< \max(e_1, f_1)$$

Therefore $j_1 = \max(e_1, f_1)$. Similarly we have

$$j_2 = \max(e_2, f_2),\ \ j_3 = \max(e_3, f_3), \cdots, j_k = \max(e_k, f_k)$$

Hence $m = p_1^{\max(e_1,\, f_1)} \times p_2^{\max(e_2,\, f_2)} \times p_3^{\max(e_3,\, f_3)} \times \cdots \times p_k^{\max(e_k,\, f_k)}$ where $[a, \ b] = m$.

■

17. We are asked to prove the following:

> Let $a = p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \times \cdots \times p_k^{e_k}$ and $b = p_1^{f_1} \times p_2^{f_2} \times p_3^{f_3} \times \cdots \times p_k^{f_k}$ be the prime decompositions of a and b and $e_j \geq 0$ and $f_j \geq 0$. Then the gcd is given by
>
> $$\gcd(a, \ b) = p_1^{\min(e_1,\, f_1)} \times p_2^{\min(e_2,\, f_2)} \times p_3^{\min(e_3,\, f_3)} \times \cdots \times p_k^{\min(e_k,\, f_k)}.$$

Proof.

Let $\gcd(a, \ b) = d$. Then d must be a product of the given primes:

$$d = p_1^{j_1} \times p_2^{j_2} \times p_3^{j_3} \times \cdots \times p_k^{j_k} \qquad \text{(some } j\text{'s may be zero)}$$

The number d *cannot* have other primes because then d would *not* be a divisor of both a and b.

We need to prove that

$$j_1 = \min(e_1, \ f_1),\ \ j_2 = \min(e_2, \ f_2),\ \ \cdots,\ \ j_k = \min(e_k, \ f_k)$$

We prove $j_1 = \min(e_1, \ f_1)$ then the others follow a very similar argument.

We consider two cases; $j_1 > \min(e_1, \ f_1)$ and $j_1 < \min(e_1, \ f_1)$. Then derive a contradiction in both cases.

<u>Case 1</u>:

Suppose $j_1 > \min(e_1, \ f_1)$. Without loss of generality assume $\min(e_1, \ f_1) = e_1$ so $j_1 > \min(e_1, \ f_1) = e_1$ or $j_1 > e_1$.

Consider

$$g = p_1^{j_1} \times p_2^{\min(e_2,\ f_2)} \times p_3^{\min(e_3,\ f_3)} \times \cdots \times p_k^{\min(e_k,\ f_k)}.$$

From this we have $p_1^{j_1} \mid g$ because $p_1^{j_1}$ is a factor of g.

Then $g \nmid a$. *Why not?*

Suppose $g \mid a$ and we already have $p_1^{j_1} \mid g$ so $p_1^{j_1} \mid a$. We are given that

$$a = p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \times \cdots \times p_k^{e_k}$$

Therefore from $p_1^{j_1} \mid a$ we have

$$p_1^{j_1} \mid \left(p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \times \cdots \times p_k^{e_k}\right)$$

All these primes p's *are distinct* so

$$\gcd\left(p_1^{j_1},\ p_2^{e_2}\right) = \gcd\left(p_1^{j_1},\ p_3^{e_3}\right) = \cdots = \gcd\left(p_1^{j_1},\ p_k^{e_k}\right) = 1$$

By the result of question 24 of the Supplementary Problems 1:

If $a \mid \left(b_1 \times b_2 \times \cdots \times b_n\right)$ and $\gcd\left(a,\ b_1\right) = \gcd\left(a,\ b_2\right) = \cdots = \gcd\left(a,\ b_{n-1}\right) = 1$

then $a \mid b_n$.

Applying this result to

$$p_1^{j_1} \mid \left(p_1^{e_1} \times p_2^{e_2} \times p_3^{e_3} \times \cdots \times p_k^{e_k}\right) \quad \text{gives} \quad p_1^{j_1} \mid p_1^{e_1}$$

This $p_1^{j_1} \mid p_1^{e_1}$ is impossible because from our supposition we have $j_1 > e_1$.

Hence $j_1 \not> \min\left(e_1,\ f_1\right)$.

<u>Case II</u>:

Suppose $j_1 < \min\left(e_1,\ f_1\right)$. Consider

$$g = p_1^{j_1} \times p_2^{\min(e_2,\ f_2)} \times p_3^{\min(e_3,\ f_3)} \times \cdots \times p_k^{\min(e_k,\ f_k)}$$

Then g is a common divisor of a and b. However

$$g \neq \gcd\left(a,\ b\right)$$

Why not?

Suppose $g = \gcd\left(a,\ b\right)$. Let us define d by

$$d = p_1^{\min(e_1,\ f_1)} \times p_2^{\min(e_2,\ f_2)} \times p_3^{\min(e_3,\ f_3)} \times \cdots \times p_k^{\min(e_k,\ f_k)}.$$

Therefore, d is also a common divisor of a and b. By Definition (1.4) (ii):

The positive integer g is the gcd of integers a and b $\Leftrightarrow$

(ii) $c \mid a$ and $c \mid b$ then $c \leq g$. [g is the largest of the common divisors]

We have $d \leq g$. This is impossible because

$$g = p_1^{j_1} \times p_2^{\min(e_2,\, f_2)} \times \cdots \times p_k^{\min(e_k,\, f_k)} \text{ and } d = p_1^{\min(e_1,\, f_1)} \times p_2^{\min(e_2,\, f_2)} \times \cdots \times p_k^{\min(e_k,\, f_k)}$$

And in our supposition we have $j_1 < \min(e_1, \ f_1)$ so $g < d$.

Hence $g \neq \gcd(a, \ b)$. Therefore $j_1 \not< \min(e_1, \ f_1)$.

Putting both of these cases together $j_1 \not> \min(e_1, \ f_1)$ and $j_1 \not< \min(e_1, \ f_1)$ we must have $j_1 = \min(e_1, \ f_1)$.

Similarly we can show that

$$j_2 = \min(e_2, \ f_2), \ j_3 = \min(e_3, \ f_3), \ \cdots, \ j_k = \min(e_k, \ f_k)$$

This completes our proof.

■

18. We are asked to prove:

If $[a, \ b] = m$ and n is a common multiple of a and b then $m \mid n$.

Proof.

Suppose $m \nmid n$. By the division algorithm we have unique integers q and r such that

$$n = mq + r \quad \text{where } 0 < r < m \qquad (*)$$

We are given that n is a common multiple of a and b so $a \mid n$ and m is a common multiple of a and b so $a \mid m$. Therefore there are integers x and x' such that

$$ax = n \text{ and } ax' = m$$

Substituting these into (*) yields

$$ax = ax'q + r \ \Rightarrow \ a(x - x'q) = r \ \Rightarrow \ a \mid r.$$

Since both m and n are common multiples of b so similarly we can show that

$$b \mid r$$

From both of these $a \mid r$ and $b \mid r$ we conclude that r is a common multiple of a and b. From (*) we have $0 < r < m$. This is a contradiction. *Why?*

Because we are given $[a, \ b] = m$ and by the definition of least common multiple we must have the common multiple r satisfying $r \geq m$ because m is the *least* common multiple.

Our supposition $m \nmid n$ must be wrong so $m \mid n$.

This completes our proof.

■

19. We are asked to prove the following:

> Let $a_1,\ a_2,\ a_3,\cdots,a_n$ be positive integers then
> $$\left[a_1,\ a_2,\ a_3,\cdots,a_n\right]=\left[\left[a_1,\ a_2,\ a_3,\cdots,a_{n-1}\right],\quad a_n\right]$$

Proof.

Let $\left[a_1,\ a_2,\ a_3,\cdots,a_n\right]=l$ and $\left[\left[a_1,\ a_2,\ a_3,\cdots,a_{n-1}\right],\quad a_n\right]=m$. Required to prove that $l=m$. *How?*

We show that $m\le l$ and then show $l\le m$. Of course this can only imply $l=m$.

<u>Case I</u>: Showing $m\le l$.

Since $\left[a_1,\ a_2,\ a_3,\cdots,a_n\right]=l$ so l is a common multiple of all the a's;

$$a_1\mid l,\ a_2\mid l,\ a_3\mid l,\cdots \text{ and } a_n\mid l$$

Therefore, l is a common multiple of $a_1,\ a_2,\ a_3,\cdots,a_{n-1}$. So l is a common multiple of

$$\left[a_1,\ a_2,\ a_3,\cdots,a_{n-1}\right].$$

Since l is a common multiple of all the a's so it is a multiple of a_n. Hence it is a common multiple of $\left[a_1,\ a_2,\ a_3,\cdots,a_{n-1}\right]$ and a_n so by the definition of the least common multiple (2.28) part (ii):

> Let m be the LCM of a and b, that is $\left[a,\quad b\right]=m$. Then m satisfies
>
> (ii) if both $a\mid l$ and $b\mid l$ then $m\le l$ - least multiple

We have $m\le l$ because $m=\left[\left[a_1,\ a_2,\ a_3,\cdots,a_{n-1}\right],\quad a_n\right]$.

<u>Case II</u>: Showing $l\le m$.

Now going the other way m is a common multiple of

$$\left[a_1,\ a_2,\ a_3,\cdots,a_{n-1}\right] \text{ and } a_{n-1}.$$

Let $\left[a_1,\ a_2,\ a_3,\cdots,a_{n-1}\right]=m'$.

Therefore m' is a common multiple of only these a's;

$$a_1\mid m',\ a_2\mid m',\ a_3\mid m',\cdots \text{ and } a_{n-1}\mid m' \qquad (*)$$

From $\left[\underbrace{\left[a_1,\ a_2,\ a_3,\cdots,a_{n-1}\right]}_{=m'},\quad a_n\right]=m$ we have the common multiple m satisfying

$$m' \mid m \text{ and } a_n \mid m$$

From (*) and this $m' \mid m$ we have

$$a_1 \mid m,\ a_2 \mid m,\ a_3 \mid m,\cdots \text{and } a_{n-1} \mid m$$

Since we have $a_n \mid m$ so m is also a common multiple of *all* the a's.

Again, by the above definition (2.28) part (ii) we have $l \leq m$.

Therefore we have $m = l$ because we have shown $m \leq l$ and $l \leq m$.

This completes our proof.

■

20. We are asked to prove $\left[n,\quad n+1\right]=n\times\left(n+1\right)$.

Proof.

First $\gcd\left(n,\quad n+1\right)=1$. *Why?*

Because we have already shown that the gcd of two consecutive integers is 1.

(See question 6(b) of Exercises 2.1.)

Now applying Proposition (2.21):

Let a and b be relatively prime positive integers then $\left[a,\quad b\right]=a\times b$.

To $a=n$ and $b=n+1$ gives

$$\left[n,\quad n+1\right]=n\times\left(n+1\right)$$

This completes our proof.

■

21. We are asked to prove that $\gcd\left(a,\quad b\right)=\gcd\left(a+b,\quad\left[a,\quad b\right]\right)$.

Proof.

Let $g=\gcd\left(a,\quad b\right)$. Then there are integers x and y such that

$$gx = a \text{ and } gy = b \qquad (*)$$

Substituting this into $\gcd\left(a+b,\quad\left[a,\quad b\right]\right)$ gives

$$\begin{aligned}\gcd\left(a+b,\ \left[a,\ b\right]\right) &= \gcd\left(gx+gy,\ \left[gx,\ gy\right]\right) \\ &= \gcd\left(g\left(x+y\right),\ g\left[x,\ y\right]\right) && \left[\begin{array}{l}\text{By result of question 7;}\\ \left[gx,\ gy\right]=g\left[x,\ y\right]\end{array}\right] \\ &= g\times\gcd\left(x+y,\ \left[x,\ y\right]\right) && \left[\begin{array}{l}\text{By Proposition (1.11);}\\ \gcd\left(ab,\ ac\right)=\left|a\right|\gcd\left(b,\ c\right)\end{array}\right]\end{aligned}$$

The $\gcd\left(x,\ y\right)=1$. *Why?*

Because by Proposition (1.5):

> If $\gcd\left(a,\ b\right)=g$ then $\gcd\left(\frac{a}{g},\ \frac{b}{g}\right)=1$

From (*) we have

$$x=\frac{a}{g} \text{ and } y=\frac{b}{g} \text{ so by Proposition (1.5) } \gcd\left(x,\ y\right)=1$$

Since $\gcd\left(x,\ y\right)=1$ so x and y are relatively prime which implies $\left[x,\ y\right]=xy$.

Substituting this $\left[x,\ y\right]=xy$ into the above derivation gives

$$\gcd\left(a+b,\ \left[a,\ b\right]\right)=g\times\gcd\left(x+y,\ \left[x,\ y\right]\right)=g\times\gcd\left(x+y,\ xy\right) \quad (\ddagger)$$

Applying the given hint:

> If $\gcd\left(x,\ y\right)=1$ then $\gcd\left(x+y,\ xy\right)=1$.

To (‡) gives

$$\gcd\left(a+b,\ \left[a,\ b\right]\right)=g\times 1=g$$

Note that $g=\gcd\left(a,\ b\right)$ so

$$\gcd\left(a+b,\ \left[a,\ b\right]\right)=g=\gcd\left(a,\ b\right)$$

This completes our proof.

■

22. We are asked to prove $\gcd\left(a,\ b,\ c\right)\times\left[ab,\ ac,\ bc\right]=a\times b\times c$.

Proof.

Using the given hint $\gcd\left(a,\ b,\ c\right)=\gcd\left(\gcd\left(a,\ b\right),\ c\right)$ and Proposition (2.23):

> $$\left[a_1,\ a_2,\ a_3,\cdots,a_n\right]=\left[\left[a_1,\ a_2,\ a_3,\cdots,a_{n-1}\right],\ a_n\right]$$

We have

$$\gcd\left(a,\quad b,\quad c\right)\times\left[ab,\quad ac,\quad bc\right]=\gcd\left(\gcd\left(a,\quad b\right),\quad c\right)\times\left[ab,\quad \left[ac,\quad bc\right]\right]$$

$$=\gcd\left(\gcd\left(a,\quad b\right),\quad c\right)\times\left[ab,\quad \underbrace{c\left[a,\quad b\right]}_{\substack{\text{By result of question 7;}\\ \left[xy,\quad xz\right]=x\times\left[y,\quad z\right]}}\right]$$

Applying Proposition (2.22):

$$\boxed{\gcd\left(x,\quad y\right)\times\left[x,\quad y\right]=xy}$$

To ab gives $\gcd\left(a,\quad b\right)\times\left[a,\quad b\right]=ab$. Putting this into the above derivation:

$$\begin{aligned}\gcd\left(a,\quad b,\quad c\right)\times\left[ab,\quad ac,\quad bc\right]&=\gcd\left(\gcd\left(a,\quad b\right),\quad c\right)\times\left[ab,\quad c\left[a,\quad b\right]\right]\\
&=\gcd\left(\gcd\left(a,\quad b\right),\quad c\right)\times\left[\gcd\left(a,\quad b\right)\times\left[a,\quad b\right],\quad c\left[a,\quad b\right]\right]\\
&\underbrace{=}_{\text{by result of question 7}}\left[a,\quad b\right]\times\left(\gcd\left(\gcd\left(a,\quad b\right),\quad c\right)\times\left[\gcd\left(a,\quad b\right),\quad c\right]\right)\\
&=\left[a,\quad b\right]\times\underbrace{\left(\gcd\left(a,\quad b\right)\times c\right)}_{\substack{\text{By (2.22) } \gcd\left(x,\quad y\right)\times\left[x,\quad y\right]=x\times y\\ \text{with } x=\gcd\left(a,\quad b\right) \text{ and } y=c}}\\
&=\underbrace{\frac{ab}{\cancel{\gcd\left(a,\quad b\right)}}}_{\text{By Proposition (2.22)}}\times\cancel{\gcd\left(a,\quad b\right)}\times c\\
&=abc\end{aligned}$$

This is our required result so it completes our proof.

■