Complete Solutions to Exercises 2.1

1. (a) We can write 56 in its prime decomposition as

$$56 = 8 \times 7 = 2^3 \times 7$$

(b) *Is 57 prime?*

No because $57 = 3 \times 19$ so it is composite and this 3×19 is the prime decomposition of 57.

(c) Earlier in the text we found $100 = 2^2 \times 5^2$. *What is the prime factors of 200?*

Multiply $100 = 2^2 \times 5^2$ by 2 which gives

$$200 = 2 \times 2^2 \times 5^2 = 2^3 \times 5^2$$

(d) *What are the prime factors of 360?*

Dividing 360 into smaller numbers and using the rules of indices we have:

$$\begin{aligned} 360 &= 36 \times 10 \\ &= 6^2 \times (2 \times 5) \\ &= (2 \times 3)^2 \times (2 \times 5) = 2^2 \times 3^2 \times 2 \times 5 = 2^3 \times 3^2 \times 5 \end{aligned}$$

(e) This number 1001 is harder to deal with. Clearly it is *not* even so 2 is not a factor of this number. *Does the next prime 3 go into 1001?*

No. [There is an easy check to see if a number is divisible by 3 – add the digits and if their (digits) sum is divisible by 3 then the initial number is also divisible by 3.]

Clearly 5 is not a factor of 1001. *What about the next prime 7?*

Yes 7 is a factor of 1001 because $7 \times 143 = 1001$. Now we need to find the factors of 143. There is no point testing the first three primes 2, 3 and 5 because if they were factors of 143 then they would be factors of 1001 which they are *not*.

The next prime 7 is not a factor of 143. *What about 11?*

11 is a factor of 143 because $11 \times 13 = 143$. Hence the prime factors of 143 are 11 and 13. Therefore we have

$$1001 = 7 \times 143 = 7 \times 11 \times 13$$

2. (a) 53 is a prime so the prime decomposition of 53 is 53.

(b) Clearly

$$530 = 10 \times 53 = 2 \times 5 \times 53$$

(c) We need to factorize 1988. Since it is even it has a factor of 2:

$$1988 = 2 \times 994$$

994 is also even so

$$1988 = 2 \times 2 \times 497$$

Now 497 is not even and if we try the next prime factor 3 we find that 3 is not a factor of 497. There is no point trying 5 because the last digit is not 5 or 0. The next prime to trial is 7:

$$497 = 7 \times 71$$

71 is a prime number so our prime decomposition of 1988 is given by

$$1988 = 2 \times 2 \times 7 \times 71 = 2^2 \times 7 \times 71$$

(d) We are given the number 666. Clearly 2 is a factor of 666 so

$$666 = 2 \times 333$$

Of course 3 is a factor of 333 so we have

$$666 = 2 \times 3 \times 111$$

3 is also a factor of 111 therefore

$$666 = 2 \times 3 \times 3 \times 37 = 2 \times 3^2 \times 37$$

Since 37 is prime so we have prime decomposition of 666 is $2 \times 3^2 \times 37$.

(e) We need to find the prime factors of 2021. It is not an even number so it does not have a factor of 2. Additionally the next prime 3 is not a factor of 2021. Clearly 5 is not a factor of 2021. The primes after 5 are

$$7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41$$

None of these are factors of 2021. However the next prime 43 is a factor because

$$2021 = 43 \times 47$$

47 is also a prime so this 43×47 is the prime decomposition of 2021.

3. (a) We are asked to prove $\gcd(a,\ p) = 1$ given that $p \nmid a$.

Proof.

Suppose $\gcd(a,\ p) = g > 1$.

Since $g \mid p$ and $g > 1$ so $g = p$ because we are given that p is prime and it only has the factors 1 and p. From the definition of gcd we have

$$g \mid a \text{ because } \gcd(a,\ p) = g$$

Therefore $p \mid a$. This is impossible because we are given $p \nmid a$. Hence our supposition $\gcd(a,\ p) = g > 1$ must be wrong so $\gcd(a,\ p) = 1$.

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(b)We are asked to prove $\gcd(p, \quad q)=1$ given p and q are *distinct* primes.

Proof.

We are given that p and q are distinct primes so $p \nmid q$. Applying the result of the previous question:

$$\boxed{\gcd(a, \quad p)=1 \text{ given that } p \nmid a}$$

With $a=q$ we have $\gcd(p, \quad q)=1$.

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4. We are asked to show that the smallest factor (larger than 1) of p^n is p.

Proof.

We can write

$$p^n = \underbrace{p \times p \times \cdots \times p}_{n \text{ copies}}$$

The factors of p^n are $p, \ p^2, \ p^3, \ \cdots, \ p^{n-1}, \ p^n$. Since p is prime (>1) so amongst this list, p is the smallest integer which is a factor of p^n. Hence we have our result.

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5. (i) We are required to prove $\gcd(p^n, \quad q^n)=1$ given p and q are distinct primes.

How do we prove this?

By contradiction.

Proof.

We are given that p and q are distinct primes so by the result of question 3 we have $\gcd(p, \quad q)=1$.

Suppose $\gcd(p^n, \quad q^n)=g>1$. Then $g \mid p^n$ and the only factors of p^n are

$$p, \ p^2, \ p^3, \ \cdots, \ p^{n-1}, \ p^n$$

Therefore g must be one of these. Without loss of generality assume

$$g=p^k \text{ where } k \text{ is an integer between 1 and } n$$

Since $p \mid p^k$ so $p \mid g$.

In the above we have $\gcd\left(p^n,\ \ q^n\right) = g$ so $g \mid q^n$. We have $p \mid g$ therefore $p \mid q^n$.

By Corollary (2.4):

If $p,\ q_1,\ q_2,\ q_3,\ \cdots,\ q_n$ are *all* primes and $p \mid q_1 \times q_2 \times q_3 \times \cdots \times q_n$ then $p = q_k$.

We have $p = q$. This is impossible because p and q are distinct primes.

Hence we have our required result by contradiction because our supposition was $\gcd\left(p^n,\ \ q^n\right) = g > 1$ which is wrong and $\gcd\left(p^n,\ \ q^n\right) = 1$.

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(ii) We need to prove that if p and q are distinct primes then $\gcd\left(p^n,\ \ q^m\right) = 1$ for any natural numbers m and n.

Proof. Like part (i).

6. (a) We are asked to prove consecutive integers have *no* prime factors in common.

Proof.

Suppose the prime p is common factor to both integers n and $n+1$. Then

$$p \mid n \text{ and } p \mid (n+1)$$

By Linear Combination Theorem (1.3):

If $a \mid b$ and $a \mid c$ then $a \mid (bx + cy)$ for any integers x and y.

We have

$$p \mid (n+1) - n \quad \Rightarrow \quad p \mid 1$$

Since p is prime so $p \nmid 1$. We have a contradiction, so our supposition that p is a common prime to n and $n+1$ is incorrect. Hence there is *no* common prime factor of two consecutive integers.

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(b) See question 18(a) of Exercises 1.1.

7. Using the product definition in each case:

(a) $\displaystyle\prod_{j=1}^{6}(2j) = 2 \times 4 \times 6 \times 8 \times 10 \times 12 = 46\ 080$

(b) $\displaystyle\prod_{j=1}^{6}\left(\frac{j}{2}\right) = \frac{1}{2} \times \frac{2}{2} \times \frac{3}{2} \times \frac{4}{2} \times \frac{5}{2} \times \frac{6}{2} = \frac{1}{2} \times 1 \times \frac{3}{\cancel{2}} \times \cancel{2} \times \frac{5}{2} \times 3 = \frac{45}{4}$

(c) Evaluating the given product $\prod_{j=1}^{3}\prod_{i=1}^{5}\left(\frac{i}{j}\right)$ is slightly more complex:

$$\begin{aligned}\prod_{j=1}^{3}\prod_{i=1}^{5}\left(\frac{i}{j}\right) &= \prod_{j=1}^{3}\left[\left(\frac{1}{j}\right)\times\left(\frac{2}{j}\right)\times\left(\frac{3}{j}\right)\times\left(\frac{4}{j}\right)\times\left(\frac{5}{j}\right)\right]\\ &= \left[\left(\frac{1}{1}\right)\times\left(\frac{2}{1}\right)\times\cdots\times\left(\frac{5}{1}\right)\right]\times\left[\left(\frac{1}{2}\right)\times\left(\frac{2}{2}\right)\times\cdots\times\left(\frac{5}{2}\right)\right]\times\left[\left(\frac{1}{3}\right)\times\left(\frac{2}{3}\right)\times\cdots\times\left(\frac{5}{3}\right)\right]\\ &= \underbrace{\left[1\times 2\times\cdots\times 5\right]}_{=120}\times\underbrace{\left[\left(\frac{1}{\cancel{2}}\right)\times(1)\times\left(\frac{3}{2}\right)\times(\cancel{2})\times\left(\frac{5}{2}\right)\right]}_{=15/4}\times\underbrace{\left[\left(\frac{1}{3}\right)\times\left(\frac{2}{3}\right)\times(1)\times\left(\frac{4}{3}\right)\times\left(\frac{5}{3}\right)\right]}_{=40/81}\\ &= 120\times\frac{15}{4}\times\frac{40}{81}=\frac{2000}{9}\end{aligned}$$

8. (a) Clearly the first part $3 \mid \left(-3\times(-5)\right)$ is correct. Of course $3 \nmid 5$ and $3 \mid (-3)$. The error is -3 is *not* a prime because from the definition of prime it has to be an integer greater than 1.

(b) *What is the error in the following:*

$$6 \nmid (2\times 5\times 7) \quad\Rightarrow\quad \gcd(6,\ 2)=\gcd(6,\ 5)=\gcd(6,\ 7)=1\,?$$

The error is $\gcd(6,\ 2)=2\neq 1$. This occurs because 6 is a composite number not a prime.

9. We are asked to show that $p,\ \ p+2$ are relatively prime.

Proof.

Suppose d is a common factor of the given integers $p,\ \ p+2$. Then

$$d \mid p, \quad\text{and}\quad d \mid p+2$$

By Linear Combination Theorem (1.3):

$$\boxed{\text{If } a \mid b \text{ and } a \mid c \text{ then } a \mid (bx+cy) \text{ for any integers } x \text{ and } y.}$$

We have $d \mid (p+2-p) \Rightarrow d \mid 2$. The only positive factors of 2 is 1 and 2. Since we are given that p is an odd prime so $d=1$. Hence $p,\ \ p+2$ have no common factor greater than 1. (They are relatively prime.)

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10. We are asked to show that one of p, $p+2$ or $p+4$ is divisible by 3.

Proof.

If the prime p is divisible by 3 then we are done. Let $3 \nmid p$ then by the Division Algorithm we have

$$p = 3q + 1 \quad \text{or} \quad p = 3q + 2$$

If $p = 3q + 1$ then $p + 2 = 3q + 1 + 2 = 3q + 3$ which implies $3 \mid (p + 2)$.

If $p = 3q + 2$ then $p + 4 = 3q + 2 + 4 = 3q + 6 = 3(q + 2)$ which implies $3 \mid (p + 4)$.

This completes our proof that one of p, $p+2$ or $p+4$ is divisible by 3.

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11. To prove a mathematical statement is false you only need to produce one counter example.

(a) The following:

If p is prime then $p + 2$ is prime.

Is *false* because let $p = 2$ then $p + 2 = 4$ which is not prime.

(b) The integer $n^2 + 1$ is not prime for $n = 8$ because

$$8^2 + 1 = 65 = 5 \times 13$$

(c) The integer $n^2 - 1$ is not composite or in other words prime only if $n = 2$:

$$2^2 - 1 = 3$$

(d) If we substitute $n = 4$ into $4n^2 - 2n + 1$ gives the composite number

$$\left(4 \times 4^2\right) - \left(2 \times 4\right) + 1 = 57 = 3 \times 19.$$

(e) $N = \left(2 \times 3 \times 5 \times 7 \times \cdots \times P\right) + 1$ is not necessarily prime because

$$N = \left(2 \times 3 \times 5 \times 7 \times 11 \times 13\right) + 1 = 30\ 031 = 59 \times 509$$

12. We are asked to prove $\sigma(p) = p + 1$ where p is a prime number.

Proof.

Let p be a prime number then the only positive factors of p are 1 and p.

Therefore $\sigma(p) = p + 1$.

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13. We are required to show that for a prime number p we have $\tau\left(p\right) = 2$.

Proof.

By the definition of prime number, we have the only factors of p are p and 1.

Hence $\tau\left(p\right) = 2$.

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