Complete Solutions to Exercises 1.3

1. We apply the Euclidean algorithm in each case.

   (a) Dividing 156 by 18 and using the division algorithm gives

$$\begin{aligned} 156 &= 8\left(18\right) + 12 \\ 18 &= 1\left(12\right) + 6 \\ 12 &= 2\left(6\right) + 0 \end{aligned}$$

   Therefore $\gcd\left(156,\ \ 18\right) = 6$.

   (b) Using the Euclidean algorithm:

$$\begin{aligned} 1011 &= 7\left(129\right) + 108 \\ 129 &= 1\left(108\right) + 21 \\ 108 &= 5\left(21\right) + 3 \\ 21 &= 7\left(3\right) + 0 \end{aligned}$$

   Hence $\gcd\left(129,\ \ 1011\right) = 3$.

   (c) We have

$$\begin{aligned} 703 &= 6\left(111\right) + 37 \\ 111 &= 3\left(37\right) + 0 \end{aligned}$$

   Therefore $\gcd\left(703,\ \ 111\right) = 37$.

   (d) Similarly we have

$$\begin{aligned} 232 &= 1\left(181\right) + 51 \\ 181 &= 3\left(51\right) + 28 \\ 51 &= 1\left(28\right) + 23 \\ 28 &= 1\left(23\right) + 5 \\ 23 &= 4\left(5\right) + 3 \\ 5 &= 1\left(3\right) + 2 \\ 3 &= 1\left(2\right) + \boxed{1} \\ 2 &= 2\left(1\right) + 0 \end{aligned}$$

   Non-zero remainder.

   Hence $\gcd\left(181,\ \ 232\right) = 1$. The numbers 181 and 132 are relatively prime.

2. The solutions to this question are not unique so you may have other answers to the ones below. However you can check your answer by substituting your x and y values into the given equation.

In each case we have same integers as question 1. We reverse the process of question 1.

(a) We had $\gcd\big(156,\ \ 18\big)=6$ so we need to solve $156x+18y=6$:

$$\begin{aligned} 156&=8\big(18\big)+12 && \big(1\big)\\ 18&=1\big(12\big)+6 && (2)\end{aligned}$$

Using the bottom line or step (2) we have

$$\begin{aligned} 6&=18-12\\ &=18-\big(156-8\big(18\big)\big) && \big[\text{By step (1)}\big]\\ &=9\big(18\big)-156=9\big(18\big)+\big(-1\big)156\end{aligned}$$

Hence the integer solution to $156x+18y=6$ is $x=-1,\ \ y=9$.

(b) By solution to question 1(b) we have $\gcd\big(129,\ \ 1011\big)=3$ so we need to solve

$$129x+1011y=3$$

By part (b) question 1 we had

$$\begin{aligned} 1011&=7\big(129\big)+108 && \big(1\big)\\ 129&=1\big(108\big)+21 && \big(2\big)\\ 108&=5\big(21\big)+3 && \big(3\big)\end{aligned}$$

Using step 3 we have

$$\begin{aligned} 3&=108-5\big(21\big)\\ &=108-5\big(129-108\big) && \big[\text{Using step (2)}\big]\\ &=6\big(108\big)-5\big(129\big) && \big[\text{Simplifying}\big]\\ &=6\big(1011-7\big(129\big)\big)-5\big(129\big) && \big[\text{Using step (1)}\big]\\ &=6\big(1011\big)-47\big(129\big) && \big[\text{Simplifying}\big]\end{aligned}$$

We have $6\big(1011\big)-47\big(129\big)=3$ which we can rewrite as

$$129\big(-47\big)+1011\big(6\big)=3$$

Hence the solution of $129x+1011y=3$ is $x=-47,\ \ y=6$.

(c) Similarly by using the solution of part (c) question 1 we have to solve

$$703x+111y=37$$

Rewriting the solution to question 1(c);

We had $703=6\big(111\big)+37$ with 37 as the subject gives

$$703-6\big(111\big)=37$$

The integer solution of $703x+111y=37$ is $x=1,\ \ y=-6$.

(d) We need to solve $181x + 232y = \gcd\left(181, \quad 232\right) = 1$. Copying the above solution of 1(d) and writing in the steps gives

$$\begin{aligned} 232 &= 1\left(181\right) + 51 && \left(1\right) \\ 181 &= 3\left(51\right) + 28 && \left(2\right) \\ 51 &= 1\left(28\right) + 23 && \left(3\right) \\ 28 &= 1\left(23\right) + 5 && \left(4\right) \\ 23 &= 4\left(5\right) + 3 && \left(5\right) \\ 5 &= 1\left(3\right) + 2 && \left(6\right) \\ 3 &= 1\left(2\right) + 1 && \left(7\right) \end{aligned}$$

First using step (7):

$$\begin{aligned} 1 &= 3 - 2 \\ &= 3 - \left(5 - 3\right) && \left[\text{Using step (6)}\right] \\ &= 2\left(3\right) - 5 \\ &= 2\left(23 - 4\left(5\right)\right) - 5 && \left[\text{Using step (5)}\right] \\ &= 2\left(23\right) - 9\left(5\right) \\ &= 2\left(23\right) - 9\left(28 - 23\right) && \left[\text{Using step (4)}\right] \\ &= 11\left(23\right) - 9\left(28\right) \\ &= 11\left(51 - 28\right) - 9\left(28\right) && \left[\text{Using step (3)}\right] \\ &= 11\left(51\right) - 20\left(28\right) \\ &= 11\left(51\right) - 20\left(181 - 3\left(51\right)\right) && \left[\text{Using step (2)}\right] \\ &= 71\left(51\right) - 20\left(181\right) \\ &= 71\left(232 - 181\right) - 20\left(181\right) && \left[\text{Using step (1)}\right] \\ &= 71\left(232\right) - 91\left(181\right) \end{aligned}$$

From this we have $71\left(232\right) - 91\left(181\right) = 1$ and we need to solve

$$181x + 232y = 1$$

Therefore $x = -91, \;\; y = 71$.

3. *How do we find the least positive integer of a linear combination?*

We use Proposition (1.10):

> (ii) The $\gcd\left(a, \quad b\right) = g$ is the least positive integer value of $ma + nb$ where m and n range over all the integers.

This means that $\gcd\left(a, \quad b\right) = g$ is the least positive integer value of a linear combination involving a and b.

(a) For $132x + 174y$ we need to find the gcd of 132 and 174. Applying the Euclidean algorithm we have:

$$\begin{aligned} 174 &= 132 + 42 \\ 132 &= 3\big(42\big) + \boxed{6} \leftarrow \text{Non-zero remainder}. \\ 42 &= 7\big(6\big) + 0 \end{aligned}$$

Hence $\gcd\big(132,\quad 74\big) = 6$ therefore the least positive integer value of the linear combination $132x + 174y$ is 6.

(b) Similarly, we need to find the gcd of 102 and 207 because we are given the linear combination $102x + 207y$. Using Euclidean algorithm:

$$\begin{aligned} 207 &= 2\big(102\big) + 3 \\ 102 &= 34\big(3\big) + 0 \end{aligned}$$

The last non-zero remainder is 3 so $\gcd\big(102,\quad 207\big) = 3$. The least positive integer value of $102x + 207y$ is 3.

(c) We need to find the least positive integer value of the given linear combination

$$99x + 1008y$$

How?

Determine the gcd of 99 and 1008. Applying the Euclidean algorithm we have

$$\begin{aligned} 1008 &= 10\big(99\big) + 18 \\ 99 &= 5\big(18\big) + \boxed{9} \leftarrow \text{Non-zero remainder} \\ 18 &= 2\big(9\big) + 0 \end{aligned}$$

Hence $\gcd\big(99,\quad 1008\big) = 9$ so the least positive integer value of $99x + 1008y$ is 9.

(d) Using the same procedure we first find the gcd of 666 and 3020 because we are examining the linear combination:

$$666x + 3020y$$

Using the Euclidean algorithm we have

$$\begin{aligned} 3020 &= 4\big(666\big) + 356 \\ 666 &= 1\big(356\big) + 310 \\ 356 &= 1\big(310\big) + 46 \\ 310 &= 6\big(46\big) + 34 \\ 46 &= 1\big(34\big) + 12 \\ 34 &= 2\big(12\big) + 10 \\ 12 &= 1\big(10\big) + \boxed{2} \end{aligned}$$

Since 2 goes into 10 so 2 is our last non-zero remainder. Hence $\gcd\left(666,\ \ 3020\right)=2$ which implies that the least positive integer value of $666x+3020y$ is 2.

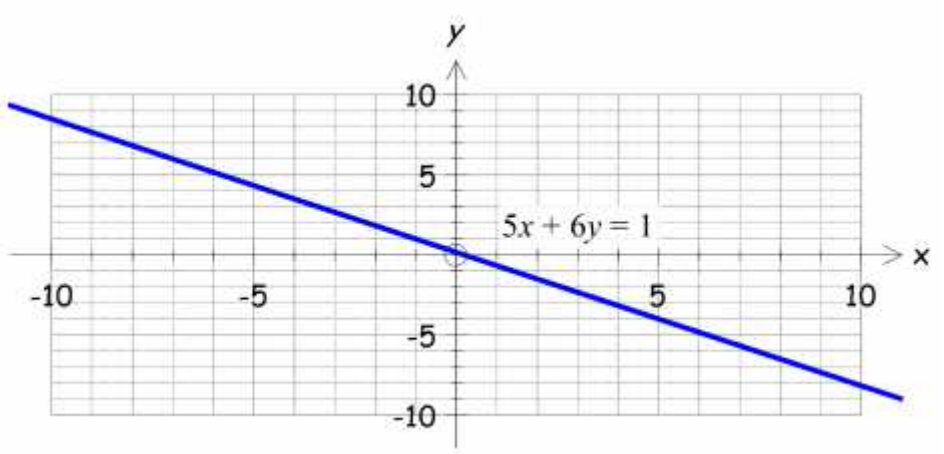
4. The solutions to this question are not unique so you may have other answers to the ones below. However you can check your answer by substituting your x and y values into the given equation.
To solve the given linear equations for integer values we can use the Euclidean algorithm to find the gcd.
(i) Applying the Euclidean algorithm for $\gcd\left(314,\ \ 785\right)$ because we want to solve
$$314x+785y=157$$
We have
$$785=2\left(314\right)+157 \qquad (*)$$
$$314=2\left(157\right)+0$$
Hence $\gcd\left(314,\ \ 785\right)=157$. Re-arranging (*) to make 157 the subject gives
$$785-2\left(314\right)=157$$
Solving $314x+785y=157$ gives $x=-2,\ \ y=1$.
(ii) *How do we solve* $314x+785y=314$ *?*
Note that $2\times157=314$ which means that 314 is double 157. We can double our solution to part (i) that is double $x=-2,\ \ y=1$ to get $x=-4,\ \ y=2$ which is our integer solution to $314x+785y=314$.
(iii) We need to solve $314x+785y=-1570$. We have the same linear combination as part (i) but the right hand side has changed to -1570 which is -10 times 157. Therefore, our solution is
$$x=-2\times\left(-10\right)=20,\ \ y=1\times\left(-10\right)=-10.$$

5. We have $ax_0+by_0=1$. Multiplying this by c gives
$$ax_0c+by_0c=c \quad \text{implies} \quad a\left(x_0c\right)+b\left(y_0c\right)=c$$
The integer solution is $x=x_0c,\ \ y=y_0c$.

6. The gcd of 20 and 28 is 4 so the least positive integer value of
$$20x+28y \text{ is } 4 \textit{ not } 2$$

Therefore, there are *no* integer solutions to $20x + 28y = 2$.

7. Sketching the graph of $5x + 6y = 1$ gives:



By examining the graph we have that when x is positive y is negative. Also when x is negative y is positive. The line does not lie in the first quadrant so there are no positive (or zero) integer solutions to $5x + 6y = 1$.

8. *Proof.*

The gcd of 198 and 5 is 1 because 5 does not go into 198 and the only positive integers going into 5 are 1 and 5. We have $\gcd(198,\ 5) = 1$.

Using Euclid's lemma (1.13):

> If $a \mid (b \times c)$ with $\gcd(a,\ b) = 1$ then $a \mid c$.

On 198 $\mid$ $5x$ yields $198 \mid x$ because $\gcd(198,\ 5) = 1$.

■

9. (a) Since we are finding integers whose gcd is 5 we only need to look for the multiples of 5:

$$a = -5,\ b = -10$$

(b) Similarly, we have $a = -100,\ b = -200$ because 100 is a divisor of both a and b. Also it is the greatest common divisor.

(c) We need to find negative integers a and b such that $\gcd(a,\ b) = 169$. Let $a = -169$ and $b = 2 \times (-169) = -338$.

10. *Proof.*

Let $g = \gcd(a, \ b)$ then $g \mid a$ and $g \mid b$. By Proposition (1.3):

> If $a \mid b$ and $a \mid c$ then $a \mid (bx + cy)$ for any integers x and y.

We have $g \mid (ax + by)$. Hence $g \mid n$ because $ax + by = n$.

■

11. Required to prove $a \mid b$ then $\gcd(a, \ b) = |a|$ where $a \neq 0$.

How?

By showing two things

1) $|a|$ is a common divisor of a and b.

2) $|a|$ is the *greatest* of the common divisors.

Proof.

<u>1) Common divisor</u>

We are given that $a \mid b$.

Clearly $|a| \mid a$ which implies $|a| \mid b$ which means that $|a|$ is a common divisor of a and b.

<u>2) Greatest</u>

We need to show $|a|$ is the greatest common divisor of a and b.

Suppose $g = \gcd(a, \ b) > |a|$. Then $g \mid a$ and because $|a| \mid a$ so $g \mid |a|$. By Proposition (1.2) part (e):

> If $x \mid y$ and $y \neq 0$ then $|x| \leq |y|$.

We have $g \leq |a|$. This is impossible because we have

$$g > |a| \text{ and } g \leq |a|$$

Our supposition $g = \gcd(a, \ b) > |a|$ must be wrong so $g = \gcd(a, \ b) = |a|$.

■

12. (i) Required to prove if $a \mid c$ and $b \mid c$, and $\gcd(a, \ b) = 1$ then $(a \times b) \mid c$.

Proof.

We are given that $a \mid c$ and $b \mid c$ so there are integers m and n such that

$$am = c \text{ and } bn = c$$

We can also assume that $\gcd(a, \ b) = 1$ so there are integers x and y such that

$$ax + by = 1$$

Multiplying this by c gives

$$axc + byc = 1 \times c = c$$

Substituting $bn = c$ for the first c on the left and $am = c$ for the second c on the left:

$$ax\left(bn\right) + by\left(am\right) = ab\left(nx + my\right) = c \quad \Rightarrow \quad ab \mid c$$

This completes our proof.

■

(ii) We are asked to prove if $a_1 \mid c,\ a_2 \mid c,\ \cdots, a_n \mid c$ and $\gcd\left(a_j,\ a_i\right) = 1$ where $i \neq j$ then

$$\left(a_1 \times a_2 \times \cdots \times a_n\right) \mid c$$

How do we prove this?

By mathematical induction.

Proof.

From part (i) we have the base case

$$a_1 \mid c,\ a_2 \mid c \text{ implies } \left(a_1 \times a_2\right) \mid c \qquad (\dagger)$$

Assume the result is true for $n = k$:

$$a_1 \mid c,\ a_2 \mid c,\ \cdots, a_k \mid c \quad \text{implies} \quad \left(a_1 \times a_2 \times \cdots \times a_k\right) \mid c \qquad (*)$$

We need to show the result for $n = k+1$ that is

$$a_1 \mid c,\ a_2 \mid c,\ \cdots, a_k \mid c,\ a_{k+1} \mid c \quad \text{implies} \quad \left(a_1 \times a_2 \times \cdots \times a_k \times a_{k+1}\right) \mid c$$

Because $\left(a_1 \times a_2 \times \cdots \times a_k\right) \times a_{k+1} = a_1 \times a_2 \times \cdots \times a_k \times a_{k+1}$ so we can rewrite the right-hand-side of the above as

$$\left(\left(a_1 \times a_2 \times \cdots \times a_k\right) \times a_{k+1}\right) \mid c$$

From (*) we have $\left(a_1 \times a_2 \times \cdots \times a_k\right) \mid c$ and we are assuming $a_{k+1} \mid c$.

Therefore by (†) we have

$$\left(\left(a_1 \times a_2 \times \cdots \times a_k\right) \times a_{k+1}\right) \mid c$$

Hence by mathematical induction we have our result.

■

13. We need to prove if $\gcd\left(a,\ b\right) = 1$ then for all d such that $d \mid a$ we have $\gcd\left(d,\ b\right) = 1$.

Proof.

Suppose $\gcd(d, \quad b) = g > 1$. Then $g \mid d$ and we are given $d \mid a$ so by Theorem (1.2) (b):

> If $a \mid b$, $b \mid c$ then $a \mid c$

We have $g \mid a$. Hence $g \mid a$ and $g \mid b$ and by the Definition (1.4) (ii):

> If for any c we have $c \mid a$ and $c \mid b$ then $c \leq g$.

we have $g \leq 1$. This $g \leq 1$ is impossible because $g > 1$. Our supposition must be wrong so $\gcd(d, \quad b) = 1$.

■

14. We need to produce a counter example to $a \mid b^2 \quad \Rightarrow \quad a \mid b$.

Let $a = 18$ and $b = 6$. Then

$$18 \mid 6^2 \quad \Rightarrow \quad 18 \not\mid 6$$

15. (i) We need to prove that

$$\gcd(a, \quad b) = \gcd(a, \quad c) = 1 \quad \Leftrightarrow \quad \gcd(a, \quad bc) = 1$$

Proof.

($\Rightarrow$). Applying Bezout's Identity (1.9):

> If $\gcd(a, \quad b) = g$ then there are integers x and y such that
> $$g = ax + by$$

To $\gcd(a, \quad b) = 1$ and $\gcd(a, \quad c) = 1$ means there are integers x, y, x' and y' such that

$$ax + by = 1 \text{ and } ax' + cy' = 1$$

Multiplying these together gives

$$\begin{aligned} (ax + by)(ax' + cy') &= 1 \\ aaxx' + acxy' + abx'y + bcyy' &= 1 \qquad [\text{Expanding}] \\ \underbrace{a(axx' + cxy' + bx'y) + bc(yy')}_{\text{Factorizing}} &= 1 \qquad (\dagger) \end{aligned}$$

The last line (†) is the least positive integer value of the linear combination of a and bc. By Proposition (1.10) part (ii):

> The $\gcd(a, \quad b) = g$ is the least positive integer value of $ma + nb$ where m and n range over all the integers.

Hence $\gcd\left(a,\quad b\times c\right)=1$.

($\Leftarrow$). Let $\gcd\left(a,\quad b\times c\right)=1$ and without loss of generality (WLOG) suppose

$$\gcd\left(a,\quad b\right)=g>1$$

By the definition of gcd we know that $g\mid a$ and $g\mid b$. Therefore

$$g\mid\left(b\times c\right)\ \text{[Because } g\mid b \text{ so } g\mid\left(b\times c\right)\text{]}$$

Now g is a common divisor of a and $b\times c$ therefore $\gcd\left(a,\quad b\times c\right)\geq g>1$. This is impossible because $\gcd\left(a,\quad b\times c\right)=1$. We have a contradiction, so $\gcd\left(a,\quad b\right)=1$. Similarly $\gcd\left(a,\quad c\right)=1$ which completes our proof. ■

(ii) We need to show that if $\gcd\left(a,\quad n_1\right)=\gcd\left(a,\quad n_2\right)=\cdots=\gcd\left(a,\quad n_k\right)=1$ then

$$\gcd\left(a,\quad n_1\times n_2\cdots\times n_k\right)=1$$

How do we prove this?

Use mathematical induction.

Proof.

We already have

$$\gcd\left(a,\quad b\right)=\gcd\left(a,\quad c\right)=1\ \text{implies}\gcd\left(a,\quad b\times c\right)=1 \qquad (\dagger)$$

By using this we have that if $\gcd\left(a,\quad n_1\right)=\gcd\left(a,\quad n_2\right)=1$ then

$$\gcd\left(a,\quad n_1\times n_2\right)=1$$

Assume the given result is true for $k=m$:

$$\gcd\left(a,\quad n_1\times n_2\cdots\times n_m\right)=1 \qquad (*)$$

We are required to prove that the result is true for $k=m+1$:

$$\gcd\left(a,\quad n_1\times n_2\cdots\times n_m\times n_{m+1}\right)=1$$

Therefore

$$\gcd\left(a,\quad n_1\times n_2\cdots\times n_m\times n_{m+1}\right)=\gcd\left(a,\quad\left(n_1\times n_2\cdots\times n_m\right)\times n_{m+1}\right)$$

From (*) we have $\gcd\left(a,\quad n_1\times n_2\cdots\times n_m\right)=1$ and we are assuming

$$\gcd\left(a,\quad n_{m+1}\right)=1$$

Now applying (†) to these $\gcd\left(a,\quad n_1\times n_2\cdots\times n_m\right)=1$ and $\gcd\left(a,\quad n_{m+1}\right)=1$ gives

$$\gcd\left(a,\quad n_1\times n_2\cdots\times n_m\times n_{m+1}\right)=\gcd\left(a,\quad \left(n_1\times n_2\cdots\times n_m\right)\times n_{m+1}\right)=1$$

Hence by mathematical induction we have our result.

■

(iii) Required to prove if $\gcd\left(a,\quad b\right)=1$ then $\gcd\left(a^n,\quad b^n\right)=1$. *How?*

Use result (i) and mathematical induction.

Proof.

We are given $\gcd\left(a,\quad b\right)=1$. The result is true for $n=1$. Assume it is true for $n=k$ that is $\gcd\left(a^k,\quad b^k\right)=1$. Required to prove this for $n=k+1$:

$$\gcd\left(a^{k+1},\quad b^{k+1}\right)=1$$

Applying the above result (i)

If $\gcd\left(a,\quad b\right)=1$ and $\gcd\left(a,\quad c\right)=1$ then $\gcd\left(a,\quad bc\right)=1$.

To $\gcd\left(a^k,\quad b^k\right)=1$ and $\gcd\left(a,\quad b\right)=1$ gives

$$\gcd\left(a^k,\quad b^kb\right)=\gcd\left(a^k,\quad b^{k+1}\right)=1$$

Repeating this process, we have

$$\gcd\left(a^{k+1},\quad b^{k+1}\right)=1$$

By mathematical induction we have our required result.

■

16. We are asked to prove that if $\gcd\left(a,\quad b\right)=1$ then $\gcd\left(a+b,\quad ab\right)=1$. *How do we prove this result?*

By contradiction.

Proof.

We are given that $\gcd\left(a,\quad b\right)=1$. Suppose $\gcd\left(a+b,\quad ab\right)=g>1$. Then

$$g\mid\left(a+b\right)\text{ and }g\mid ab.$$

We also have g is a divisor of a linear combination of these, $a+b$ and ab.

Therefore

$$g\mid\left(a+b\right)a-ab\ \Rightarrow\ g\mid a^2$$

Similarly by another linear combination of $a+b$ and ab we have

$$g\mid\left(a+b\right)b-ab\ \Rightarrow\ g\mid b^2$$

Hence $g>1$ is a common factor of a^2 and b^2. By the definition of gcd we have

$$\gcd\left(a^2, \quad b^2\right) \geq g > 1$$

This is impossible because by the result of the previous question 15(ii) we have with $n = 2$:

$$\gcd\left(a^2, \quad b^2\right) = 1$$

We have a contradiction, so our supposition $\gcd\left(a+b, \quad ab\right) = g > 1$ must be wrong and $\gcd\left(a+b, \quad ab\right) = 1$. This completes our proof.

■

17. We are asked to prove $\gcd\left(ma, \quad mb\right) = mg$ where $\gcd\left(a, \quad b\right) = g$.

Proof.

By Bezout's Identity (1.9) there exist integers x and y such that

$$ax + by = \gcd\left(a, \quad b\right) = g$$

By applying Proposition (1.10) part (ii) we have

$$\begin{aligned}\gcd\left(ma, \quad mb\right) &= \text{least positive value of } m\left(ax\right) + m\left(by\right) \\ &= m \ \ (\text{least positive value of } ax + by\,) = mg\end{aligned}$$

■

18. We are asked to prove that if $g = \gcd\left(a, \quad b\right)$ and $d \mid a$, $d \mid b$ then $d \mid g$.

Proof.

By Bezout's Identity (1.9) there exist integers x and y such that

$$ax + by = \gcd\left(a, \quad b\right) = g \quad (*)$$

We are also given that $d \mid a$ and $d \mid b$ so there are integers m and n such that

$$dm = a \ \text{ and } \ dn = b$$

Substituting these $dm = a$ and $dn = b$ into (*) gives

$$ax + by = dmx + dny = d\left(mx + ny\right) = g \quad \Rightarrow \ d \mid g$$

This completes our proof.

■

19. We need to prove that

$$\gcd\left(a, \quad b, \quad c\right) = \gcd\left(a, \quad \gcd\left(b, \quad c\right)\right)$$

Proof.

Let $g = \gcd\left(a, \quad b, \quad c\right)$ and $d = \gcd\left(a, \quad \gcd\left(b, \quad c\right)\right)$. We prove that $g = d$.

Since d is a common divisor of a and $\gcd\left(b, \quad c\right)$, or in symbolic form

$$d \mid a \text{ and } d \mid \gcd\left(b, \quad c\right)$$

Since $d \mid \gcd\left(b, \quad c\right)$ and of course $\gcd\left(b, \quad c\right) \mid b$ and $\gcd\left(b, \quad c\right) \mid c$ because $\gcd\left(b, \quad c\right)$ is a common divisor of b and c.

By Theorem (1.2) (b):

If $x \mid y$ and $y \mid z$ then $x \mid z$

we have $d \mid b$ and $d \mid c$. Hence d is a common divisor of a, b and c. By the definition of gcd, $d \leq g$.

Similarly we have $g = \gcd\left(a, \quad b, \quad c\right)$ so $g \mid a$, $g \mid b$ and $g \mid c$ therefore by the result of previous question we have $g \mid \gcd\left(b, \quad c\right)$. We have

$$g \mid a \text{ and } g \mid \gcd\left(b, \quad c\right).$$

By definition of gcd $g \leq d$. As we have $d \leq g$ and $g \leq d$ therefore $g = d$.

This completes our proof.

■