Complete Solutions to Exercise 2.2

1. (a) We are given $\lfloor 5 \rfloor$ which is the floor function of 5. Clearly $\lfloor 5 \rfloor = 5$.

   (b) Although 5.999 is closer to 6 but the floor function of 5.999 is given by
$$\lfloor 5.999 \rfloor = 5$$

   (c) Evaluating $\pi^e = 22.459$ (3dp) so
$$\lfloor \pi^e \rfloor = \lfloor 22.459 \rfloor = 22$$

   (d) Similarly, we have $\lfloor e^\pi \rfloor = \lfloor 23.14 \rfloor = 23$.

   (e) This time, we are given the ceiling function $\lceil 7 \rceil$. Therefore
$$\lceil 7 \rceil = 7$$

   (f) Now we have to evaluate $\lceil 7.0000000001 \rceil$. Although $7.0000000001 \simeq 7$ but because we are dealing with the ceiling function so
$$\lceil 7.0000000001 \rceil = 8$$

   (g) Similar to part (c) but this time we are given the ceiling function so
$$\lceil \pi^e \rceil = \lceil 22.459 \rceil = 23$$

   (h) We have
$$\lceil e^\pi \rceil = \lceil 23.14 \rceil = 24$$

2. (a) We are asked to compute $\lfloor 6.3 \rfloor + \lceil -6.3 \rceil$:
$$\lfloor 6.3 \rfloor + \lceil -6.3 \rceil = 6 + (-6) = 0$$

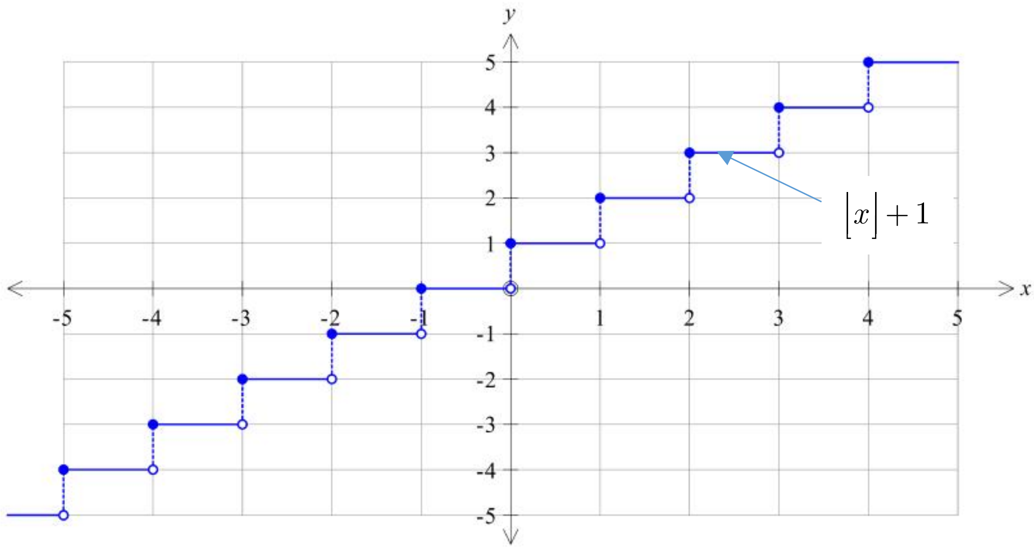
   (b) This time, we need to find $\lfloor 6.3 \rfloor + \lfloor -6.3 \rfloor$:
$$\lfloor 6.3 \rfloor + \lfloor -6.3 \rfloor = 6 + (-7) = -1$$

   (c) Similarly, we have
$$\lfloor -6.3 \rfloor + \lceil -6.3 \rceil = -7 + (-6) = -13$$

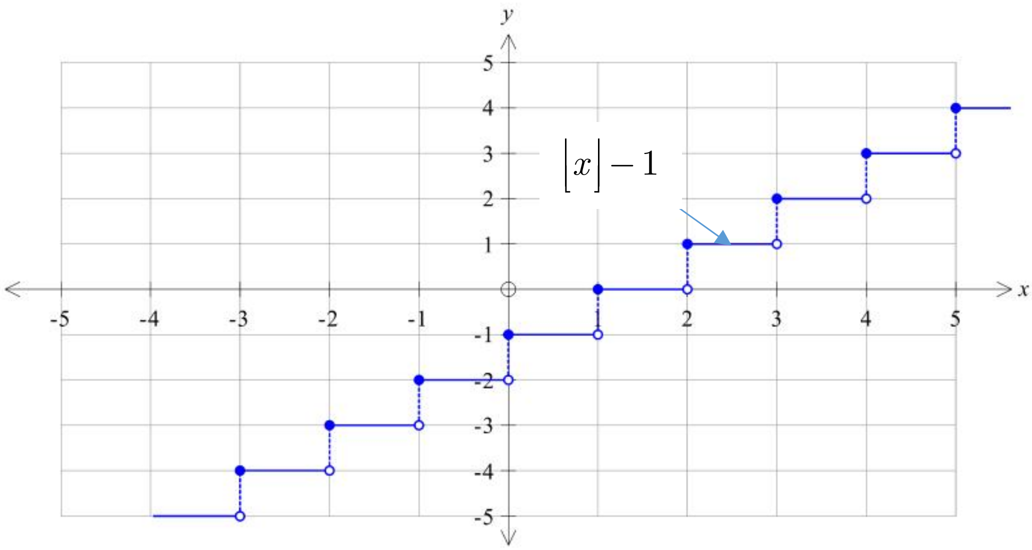
   (d) We have to compute
$$\lfloor -6.3 \rfloor + \lfloor -6.3 \rfloor = -7 + (-7) = -14$$

3. For $\lfloor x \rfloor = \lceil x \rceil$ any integer x because
$$\lfloor \text{integer} \rfloor = \text{integer} = \lceil \text{integer} \rceil$$

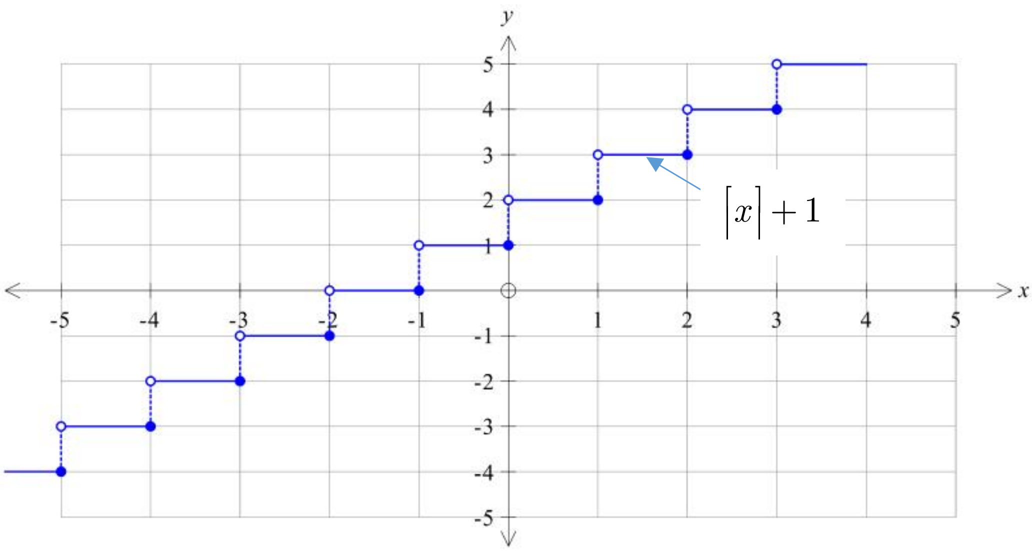
4. (a) The graph of $\lfloor x \rfloor + 1$ is very similar to the graph of $\lfloor x \rfloor$ but shifted up by one unit:



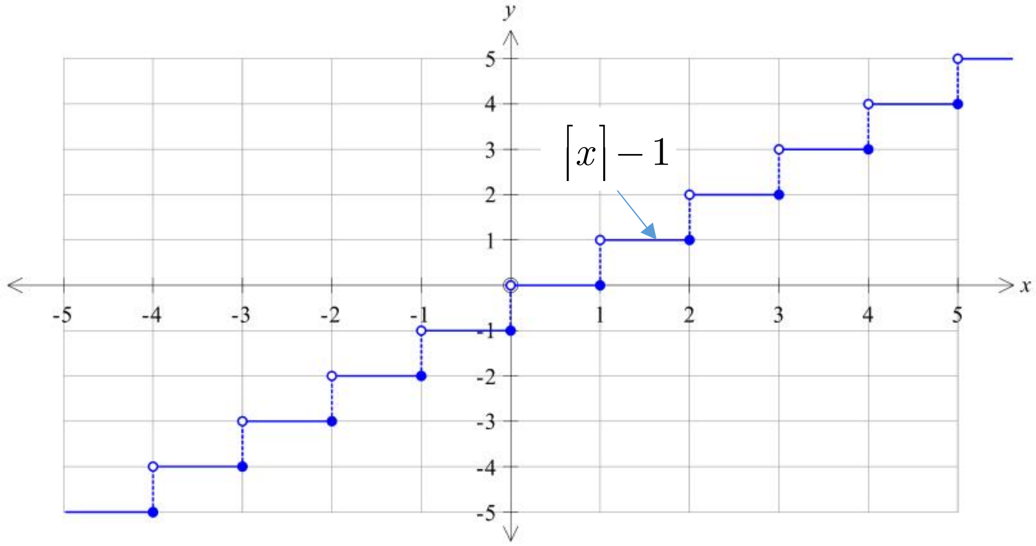
(b) Similarly we have the graph of $\lfloor x \rfloor - 1$:



(c) We are asked to plot the graph of $\lceil x \rceil + 1$ which is the graph of $\lceil x \rceil$ but shifted up by one unit:



(d) Similarly, we have the graph of $\lceil x \rceil - 1$:



5. (a) We are asked to show $\lfloor 2 \times x \rfloor = 2 \times \lfloor x \rfloor$ is false. This means we have to produce a counter example. Let $x = -6.3$ then

$$\lfloor 2 \times x \rfloor = \lfloor 2 \times (-6.3) \rfloor = \lfloor -12.6 \rfloor = -13$$
$$2 \times \lfloor x \rfloor = 2 \times \lfloor -6.3 \rfloor = 2 \times (-7) = -14$$

Hence $\lfloor 2 \times x \rfloor \neq 2 \times \lfloor x \rfloor$ [Not Equal].

(b) To show that $\lceil 2 \times x \rceil = 2 \times \lceil x \rceil$ is false we need an example where this statement is not true. Let $x = 6.3$ then

$$\lceil 2 \times x \rceil = \lceil 2 \times 6.3 \rceil = \lceil 12.6 \rceil = 13$$
$$2 \times \lceil x \rceil = 2 \times \lceil 6.3 \rceil = 2 \times 7 = 14$$

Therefore $\lceil 2 \times x \rceil \neq 2 \times \lceil x \rceil$ [Not Equal].

(c) We are asked to show $\lceil x \rceil \neq \lfloor x \rfloor + 1$. Let $x = 2$ (could be any integer) then

$$\lceil x \rceil = \lceil 2 \rceil = 2 \quad \text{but} \quad \lfloor x \rfloor + 1 = \lfloor 2 \rfloor + 1 = 3$$

Thus $\lceil x \rceil \neq \lfloor x \rfloor + 1$.

6. We are given $n - 1 < x < n$ where n is a natural number.

(a) We need to prove that $\lceil x \rceil = n$.

Proof.

By the definition of ceiling function (2.8):

$$\boxed{\lceil x \rceil = \min\{n : \ n \geq x, \ \text{integer } n\}}$$

We can rewrite the given $n-1<x<n$ as $n>x>n-1$. By applying this definition we have

$$\lceil x\rceil = \min\{n:\ n\ge x,\ \text{integer } n\} = \min\{n,\ n+1,\ n+2,\ \cdots\} = n$$

This completes our proof.

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(b) This time, we have to prove $\lfloor x\rfloor = n-1$.

Proof.

By the definition of the floor function (2.7):

$$\boxed{\lfloor x\rfloor = \max\{n:\ n\le x,\ \text{integer } n\}}$$

From the given $n-1<x<n$ and considering the left inequality we have

$$\lfloor x\rfloor = \max\{n:\ n\le x,\ \text{integer } n\} = \max\{\cdots,\ n-3,\ n-2,\ n-1\} = n-1$$

Note that n is *not* a member of this set because we are given $x<n$ or $n>x$.

This completes our proof.

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7. (a) We need to show $\lfloor x\rfloor = n$ provided $x-1<n<x$.

Proof.

By the definition of the floor function (2.7):

$$\boxed{\lfloor x\rfloor = \max\{n:\ n\le x,\ \text{integer } n\}}$$

We are given the inequality $x-1<n<x$. Adding 1 to the left-hand inequality gives

$$x<n+1$$

From the given inequality on the right-hand we have $n<x$. Combining these, $n<x$ and $x<n+1$, together yields

$$n<x<n+1$$

Applying the floor function definition to this inequality $n<x<n+1$:

$$\begin{aligned}\lfloor x\rfloor &= \max\{n:\ n\le x,\ \text{integer } n\}\\ &= \max\{\cdots, n-2,\ n-1,\ n\} = n\end{aligned}$$

This completes our proof.

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(b) This time we are asked to show $\lceil x\rceil = n+1$ given $x-1<n<x$.

Proof.

We need to use the definition of ceiling function (2.8):

$$\lceil x \rceil = \min\{n : \ n \geq x, \ \text{integer } n\}$$

Since we are given $x-1<n<x$ so n is less than x, which implies that the next integer $n+1$ is greater than x. *Why?*

Adding 1 to the given inequality $x-1<n<x$ on the left-hand-side gives

$$x<n+1$$

The right-hand inequality is $n<x$. Combining our results we have

$$n<x<n+1 \text{ or other way } n+1>x>n$$

Applying the ceiling function definition (2.8) we have

$$\begin{aligned}\lceil x \rceil &= \min\{n : \ n \geq x, \ \text{integer } n\}\\ &= \min\{n+1, \ n+2, \ \cdots\} = n+1\end{aligned}$$

This completes our proof.

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8. We use corollary (2.10) in each case:

If $n>1$ is composite then it has a prime divisor p such that $p \leq \lfloor \sqrt{n} \rfloor$.

(a) By using this corollary with $n=161$ we have

$$\lfloor \sqrt{161} \rfloor = \lfloor 12.69 \rfloor = 12$$

The prime numbers below 12 are 2, 3, 5, 7 and 11. Clearly 2, 3 and 5 are *not* factors of 161. However 7 is a factor of 161 because

$$\frac{161}{7} = 23 \quad \text{or} \quad 161 = 7 \times 23$$

(b) We need to test 203 for compositeness:

$$\lfloor \sqrt{203} \rfloor = \lfloor 14.24780685 \rfloor = 14$$

The prime factors below 14 are 2, 3, 5, 7, 11 and 13.

We can easily check that 2, 3 and 5 are not factors of 203. But

$$\frac{203}{7} = 29 \quad \text{or} \quad 203 = 7 \times 29$$

(c) We need to check that 1003 is composite or not:

$$\lfloor \sqrt{1003} \rfloor = \lfloor 31.67017524 \rfloor = 31$$

The prime factors below 31 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 and 31.

Trialing these prime factors we find that 17 is a factor of 1003 because

$$\frac{1003}{17}=59 \text{ or } 1003=17\times 59$$

You can check that 59 is prime as well. Hence the prime decomposition of 1003 is 17×59.

(d) We are given the integer 1009. Similarly we have

$$\left\lfloor\sqrt{1009}\right\rfloor=\left\lfloor 31.76476035\right\rfloor=31$$

We trial the same prime factors as part (c). We find that none of these are factors of 1009 therefore 1009 is a prime.

9. (a) We are given $(2\times 3\times 5\times 7)-1$ which is equal to 209. Clearly 2, 3, 5 and 7 cannot be factors of this number $(2\times 3\times 5\times 7)-1$ because dividing by these numbers leaves a remainder of -1. We apply Corollary (2.10):

If $n>1$ is composite then it has a prime divisor p such that $p\le\left\lfloor\sqrt{n}\right\rfloor$.

If we have a prime factor p then it must satisfy $p\le\left\lfloor\sqrt{209}\right\rfloor=14$. So the only primes left which are less than 14 are 11 and 13. We find that

$$\frac{209}{11}=19 \text{ or } 209=11\times 19$$

(b) This time we are given the number $(2\times 3\times 5\times 7)+1=211$. Again 2, 3, 5 and 7 *cannot* be factors of 211. Also the prime factors less than $\left\lfloor\sqrt{211}\right\rfloor=14$ are 11 and 13. We find that 211 *cannot* by divided by 11 and 13. By the contrapositive of Corollary (2.10) we conclude that 211 is prime.

10. We need to test the number $(2\times 3\times 5\times 7\times 11\times 13)+1=30\ 031$ for compositeness. If 30 031 is composite then it must have a prime factor p which satisfies

$$p\le\left\lfloor\sqrt{30031}\right\rfloor=\left\lfloor 173.2945469\right\rfloor=173$$

One prime factor of 30 031 must be less than or equal to 173. Clearly 2, 3, 5, 7, 11 and 13 *cannot* be factors of 30 031 because

$$(2\times 3\times 5\times 7\times 11\times 13)+1=30\ 031$$

We try the primes after this; 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, …
The last prime 59 in this list is actually a factor of 30 031 because

$$\frac{30031}{59} = 509 \quad \text{or} \quad 30031 = 59 \times 509$$

Since we have found a factor of 30 031 so it is composite. (The prime decomposition is 59×509 because 509 is also prime.)

11. We are asked to show that $2^{3^n} + 1$ is composite.

Proof.

We use the following identity given in the hint to prove this:

$$x^m + 1 = (x+1)\left(x^{m-1} - x^{m-2} + x^{m-3} - \cdots + x^2 - x + 1\right) \text{ provided } m \text{ is odd}$$

Clearly $m = 3^n$ is odd so applying this to the given integer $2^{3^n} + 1$ yields

$$2^{3^n} + 1 = (2+1)\left(2^{3^n-1} - 2^{3^n-2} + 2^{3^n-3} - \cdots + 2^2 - 2 + 1\right)$$

Since $2 + 1 = 3$ so 3 is a factor of $2^{3^n} + 1$. We also need to show that the other factor is greater than 1, that is

$$\left(2^{3^n-1} - 2^{3^n-2} + 2^{3^n-3} - \cdots + 2^2 - 2 + 1\right) > 1$$

How?

Well the left – hand side $2^{3^n} + 1 > 3$ for every natural number n. *Why?*

Because $2^{3^n} > 2$ for every natural number n.

Thus, for all n we have $2^{3^n} + 1$ is composite.

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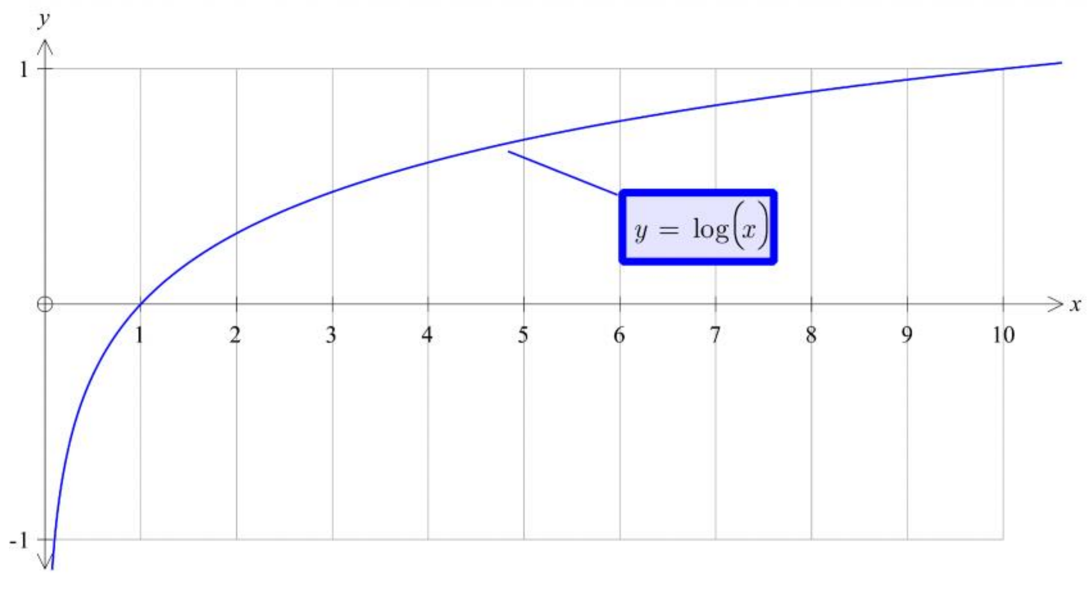
12. First note that $\log_{10}(x)$ is an increasing function, this means as we increase x so $\log_{10}(x)$ increases as you can see from the graph below:



(a) We need to find $\lceil \log_{10}(101) \rceil$ without using a calculator. From the properties of logs we have

$$\log_{10}(100) = \log_{10}(10^2) = 2 \text{ and } \log_{10}(1000) = \log_{10}(10^3) = 3$$

Since 101 lies between 100 and 1000 therefore

$$2 < \lceil \log_{10}(101) \rceil < 3$$

The ceiling function is given by:

(2.8)	$\lceil x \rceil = \min\{n : \ n \ge x, \ \text{integer } n\}$

By this definition we have $\lceil \log_{10}(101) \rceil$ is an integer which is greater than or equal to $\log_{10}(101)$ so $\lceil \log_{10}(101) \rceil = 3$.

(b) We are asked to find $\lfloor \log_2(63) \rfloor$. Using the properties of logs we have

$$\log_2(32) = \log_2(2^5) = 5 \text{ and } \log_2(64) = \log_2(2^6) = 6$$

Because 63 lies between 32 and 64 so $5 < \log_2(63) < 6$. The question says we need to find the floor function of this $\log_2(63)$ so $\lfloor \log_2(63) \rfloor = 5$.

(c) We need to find $\lfloor \log_n(n^x) \rfloor$ given $n-1 < x < n$. Using log properties

$$\lfloor \log_n(n^x) \rfloor = \lfloor x \log_n(n) \rfloor = \lfloor x \rfloor \qquad [\text{because } \log_n(n) = 1]$$

Since $n-1 < x < n$ so the floor function of x is $n-1$ (see question 6(b)). We have

$$\lfloor \log_n(n^x) \rfloor = n-1$$

13. (a) We are asked to show $\lfloor \log_{10}(N) \rfloor + 1$ gives the number of digits of $N \ge 1$.

Proof.

Let $N = a_n a_{n-1} \cdots a_1 a_0$ where the a's are the digits of N and a_n is non-zero. (The actual number of digits of N is $n+1$ because our right-hand digit is a_0).

We can rewrite this as

$$N = a_n a_{n-1} \cdots a_1 a_0 = (0 \cdot a_n a_{n-1} \cdots a_1 a_0) \times 10^{n+1}$$

Because shifting the decimal point in $0 \cdot a_n a_{n-1} \cdots a_1 a_0$ by $n+1$ places to the right gives $a_n a_{n-1} \cdots a_1 a_0 \cdot 0 = a_n a_{n-1} \cdots a_1 a_0 = N$.

Taking logs of both sides gives

$$\begin{aligned}
\log_{10}\left(N\right) &= \log_{10}\left(0\cdot a_n a_{n-1}\cdots a_1 a_0 \times 10^{n+1}\right)\\
&= \log_{10}\left(0\cdot a_n a_{n-1}\cdots a_1 a_0\right) + \log\left(10^{n+1}\right) \quad \left[\text{Applying } \log\left(A\times B\right) = \log\left(A\right)+\log\left(B\right)\right]\\
&= \log_{10}\left(0\cdot a_n a_{n-1}\cdots a_1 a_0\right) + \left(n+1\right)\log\left(10\right) \quad \left[\text{Applying } \log\left(A^n\right) = n\log\left(A\right)\right]\\
&= \log_{10}\left(0\cdot a_n a_{n-1}\cdots a_1 a_0\right) + \left(n+1\right) \quad \left[\text{Because } \log\left(10\right)=1\right]
\end{aligned}$$

Now $\log\left(0\cdot a_n a_{n-1}\cdots a_1 a_0\right)$ lies between -1 and 0, that is

$$-1 \le \log_{10}\left(0\cdot a_n a_{n-1}\cdots a_1 a_0\right) < 0$$

Why?

Because a_n is non-zero so $0\cdot a_n a_{n-1}\cdots a_1 a_0$ lies strictly between 0 and 1 (cannot equal 0 or 1) . Substituting this inequality

$$-1 \le \log_{10}\left(0\cdot a_n a_{n-1}\cdots a_1 a_0\right) < 0$$

into the above

$$\log_{10}\left(N\right) = \underbrace{\log_{10}\left(0\cdot a_n a_{n-1}\cdots a_1 a_0\right)}_{-1\le\log_{10}\left(0\cdot a_n a_{n-1}\cdots a_1 a_0\right)<0} + \left(n+1\right)$$

Gives

$$-1+\left(n+1\right) = n \le \log_{10}\left(N\right) \text{ and } \log_{10}\left(N\right) < 0+\left(n+1\right) = n+1$$

Putting these together

$$n \le \log_{10}\left(N\right) < n+1$$

By the definition of the floor function and result of question 6 part (b) we have

$$\left\lfloor \log_{10}\left(N\right)\right\rfloor = n$$

Note that $N = a_n a_{n-1}\cdots a_1 a_0$ has $n+1$ digits so $\left\lfloor \log_{10}\left(N\right)\right\rfloor + 1$ gives the number of digits of N.

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(b) The number of digits of Googol $= 10^{100}$ can be found using the result of part (a) $\left\lfloor \log_{10}\left(N\right)\right\rfloor + 1$. Let $N = 10^{100}$ then

$$\begin{aligned}
\left\lfloor \log_{10}\left(10^{100}\right)\right\rfloor + 1 &= \left\lfloor 100\times\log_{10}\left(10\right)\right\rfloor + 1 \quad \left[\text{By } \log\left(A^n\right) = n\times\log\left(A\right)\right]\\
&= \left\lfloor 100\times 1\right\rfloor + 1 = 100+1 = 101
\end{aligned}$$

The term Googol has 101 digits.

(c) The number of digits in $10^{\left(10^{100}\right)}$ can be found similarly to part (b):

$$\begin{aligned}
\left\lfloor \log_{10}\left(10^{\left(10^{100}\right)}\right)\right\rfloor + 1 &= \left\lfloor 10^{100}\times\log_{10}\left(10\right)\right\rfloor + 1 \quad \left[\text{By } \log\left(A^n\right) = n\times\log\left(A\right)\right]\\
&= \left\lfloor 10^{100}\right\rfloor + 1 = 10^{100}+1
\end{aligned}$$

The number of digits in Googolplex is $10^{100}+1$.

(d) (i) We are asked to find the number of digits of $2^{74\,207\,211}$. Taking log of this gives

$$\log_{10}\left(2^{74\,207\,211}\right)$$

Trying to evaluate this on our calculator gives an error. *How can we find this?* Convert to the base 2 by using the given hint:

$$\log_{10}\left(2^{74\,207\,211}\right)=\frac{\log_2\left(2^{74\,207\,211}\right)}{\log_2\left(10\right)}=\frac{74\,207\,211}{3.322}=22\,338\,596.41$$

Taking the floor and adding one gives the number of digits

$$\left\lfloor\log_{10}\left(2^{74\,207\,211}\right)\right\rfloor+1=\left\lfloor 22\,338\,596.41\right\rfloor+1=22\,338\,596+1=22\,338\,597$$

(ii) *How many digits does* $2^{74\,207\,211}$ *have in base 2 number system?*

Well using log to the base 2 we have

$$\log_2\left(2^{74\,207\,211}\right)+1=\left\lfloor 74\,207\,211\times\underbrace{\log_2\left(2\right)}_{=1}\right\rfloor+1=74\,207\,211+1=74\,207\,212$$

14. (i) We need to show that $\sqrt{\lfloor x\rfloor}=\left\lfloor\sqrt{x}\right\rfloor$ is false. Consider $x=12.5$ (any positive non – integer will do):

$$\sqrt{\lfloor x\rfloor}=\sqrt{\lfloor 12.5\rfloor}=\sqrt{12}=3.464\ (3\text{dp})$$
$$\left\lfloor\sqrt{x}\right\rfloor=\left\lfloor\sqrt{12.5}\right\rfloor=\lfloor 3.464\rfloor=3$$

Hence $\sqrt{\lfloor x\rfloor}\neq\left\lfloor\sqrt{x}\right\rfloor$ [Not equal].

(ii) We are asked to prove $\left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor=\left\lfloor\sqrt{x}\right\rfloor$ for $x\geq 0$.

Proof.

Clearly if x is a non-negative integer then the given result holds because $\lfloor x\rfloor=x$.

Let $\left\lfloor\sqrt{x}\right\rfloor=n$ where n is a natural number.

If x is positive real number but not an integer then there exists positive integer n such that $n-1<x<n$. Then by the result of question 6(b) we have $\lfloor x\rfloor=n-1$. We have

$$\left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor=\left\lfloor\sqrt{n-1}\right\rfloor \qquad (*)$$

By applying the following square root property for $a \geq 0,\ b \geq 0$:

$$\sqrt{a} \leq \sqrt{b} \quad \Leftrightarrow \quad a \leq b$$

To $n-1 < x < n$ we have $\sqrt{n-1} < \sqrt{x} < \sqrt{n}$.

By using the definition of floor function (2.7):

$$\boxed{\lfloor x \rfloor = \max\left\{m :\ m \leq x,\ \text{ integer } m\right\}}$$

On this $\left\lfloor \sqrt{x} \right\rfloor$ gives

$$\begin{aligned} \left\lfloor \sqrt{x} \right\rfloor &= \max\left\{m :\ m \leq \sqrt{x},\ \text{ integer } m\right\} \\ &= \max\left\{\cdots \left\lfloor \sqrt{n-3} \right\rfloor,\ \left\lfloor \sqrt{n-2} \right\rfloor,\ \left\lfloor \sqrt{n-1} \right\rfloor\right\} = \left\lfloor \sqrt{n-1} \right\rfloor \end{aligned}$$

By (*) we have $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \left\lfloor \sqrt{n-1} \right\rfloor$ therefore

$$\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \left\lfloor \sqrt{n-1} \right\rfloor \underbrace{=}_{\text{From above}} \left\lfloor \sqrt{x} \right\rfloor$$

This completes our proof.

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