Complete Solutions to Exercises 4.5

1. The Lucas – Lehmer Test is given by:

> The Mersenne number $M_p = 2^p - 1$ is prime $\Leftrightarrow$ $S_{p-2} \equiv 0 \pmod{M_p}$ where S_k is defined as the least non – negative residue such that
>
> $$S_0 = 4$$
>
> $$S_k \equiv S_{k-1}^2 - 2 \pmod{M_p} \text{ for integer } k \geq 1.$$

We are asked test $M_{13} = 2^{13} - 1 = 8191$ for primality.

Substituting $p = 13$, $M_{13} = 8191$ and $k = 1, 2, 3, \cdots, 11$ into the above formula gives

$$S_1 \equiv S_0^2 - 2 \equiv 4^2 - 2 \equiv 14 \pmod{8191}$$

$$S_2 \equiv S_1^2 - 2 \equiv 14^2 - 2 \equiv 194 \pmod{8191}$$

$$S_3 \equiv S_2^2 - 2 \equiv 194^2 - 2 \equiv 4870 \pmod{8191}$$

$$S_4 \equiv S_3^2 - 2 \equiv 4870^2 - 2 \equiv 3953 \pmod{8191}$$

$$S_5 \equiv S_4^2 - 2 \equiv 3953^2 - 2 \equiv 5970 \pmod{8191}$$

$$S_6 \equiv S_5^2 - 2 \equiv 5970^2 - 2 \equiv 1857 \pmod{8191}$$

$$S_7 \equiv S_6^2 - 2 \equiv 1857^2 - 2 \equiv 36 \pmod{8191}$$

$$S_8 \equiv S_7^2 - 2 \equiv 36^2 - 2 \equiv 1294 \pmod{8191}$$

$$S_9 \equiv S_8^2 - 2 \equiv 1294^2 - 2 \equiv 3470 \pmod{8191}$$

$$S_{10} \equiv S_9^2 - 2 \equiv 3470^2 - 2 \equiv 128 \pmod{8191}$$

$$S_{11} \equiv S_{10}^2 - 2 \equiv 128^2 - 2 \equiv 0 \pmod{8191}$$

Since $S_{11} \equiv 0 \pmod{8191}$ so by the Lucas – Lehmer test we conclude that

$$M_{13} = 2^{13} - 1 = 8191 \text{ is prime.}$$

(We have already shown M_{13} is prime by trying to find primes factors of M_{13}. See question 6(a) of Exercises 4.4.)

2. We need to find the sigma function for each of the given positive integers. We use the multiplicative property of the sigma function to find $\sigma(n)$.

(a) The prime decomposition of 15 is

$$15 = 3 \times 5.$$

Using the multiplicative property of $\sigma(n)$ we have

$$\sigma(15) = \sigma(3 \times 5) = \sigma(3) \times \sigma(5) \qquad (*)$$

How do we find $\sigma(3)$ and $\sigma(5)$?

Since 3 and 5 are primes so we use Proposition (4.32):

$$\boxed{\sigma(p) = p + 1}$$

Therefore $\sigma(3) = 4$ and $\sigma(5) = 6$. Substituting this into (*) yields

$$\sigma(15) = \sigma(3) \times \sigma(5) = 4 \times 6 = 24.$$

We can check $\sigma(15) = 24$ because this result means that the divisors of 15 add up to 24:

$$\sigma(15) = 1 + 3 + 5 + 15 = 24$$

(b) This time we have to find $\sigma(77)$. Using the same procedure as outlined in part (a) we have

$$\begin{aligned}\sigma(77) &= \sigma(7 \times 11) \\ &= \sigma(7) \times \sigma(11) = 8 \times 12 = 96\end{aligned}$$

(c) Since the sum of the digits of 171 is given by

$$1 + 7 + 1 = 9 \text{ and } 9 \mid 9$$

Therefore 9 is a factor of 171. Hence

$$171 = 9 \times 19 = 3^2 \times 19$$

Using the multiplicative property of the sigma function we have

$$\sigma(171) = \sigma(3^2 \times 19) = \sigma(3^2) \times \sigma(19) \qquad (*)$$

Using Proposition (4.35):

$$\boxed{\sigma(p^k) = \frac{p^{k+1} - 1}{p - 1}}$$

We have

$$\sigma(3^2) = \frac{3^{2+1} - 1}{3 - 1} = \frac{26}{2} = 13.$$

Applying Proposition (4.32):

$$\boxed{\sigma(p) = p + 1}$$

on $\sigma(19)$ gives

$$\sigma(19) = 19 + 1 = 20\,.$$

Substituting these results $\sigma(3^2) = 13$ and $\sigma(19) = 20$ into (*) gives

$$\sigma(171) = 13 \times 20 = 260$$

The sum of *all* the factors of 171 is 260.

(d) We are asked to evaluate $\sigma(200)$. Factorizing 200 gives

$$200 = 8 \times 25 = 2^3 \times 5^2$$

Applying the multiplicative property of the sigma function we have

$$\begin{aligned}\sigma(200) &= \sigma(2^3 \times 5^2) \\ &= \sigma(2^3) \times \sigma(5^2) \qquad (\dagger)\end{aligned}$$

Using Proposition (4.35) on each of these:

$$\boxed{\sigma(p^k) = \frac{p^{k+1} - 1}{p - 1}}$$

Gives

$$\sigma(2^3) = \frac{2^{3+1} - 1}{2 - 1} = 15 \text{ and } \sigma(5^2) = \frac{5^3 - 1}{5 - 1} = \frac{124}{4} = 31\,.$$

Substituting these calculations into $(\dagger)$ yields

$$\sigma(200) = \sigma(2^3) \times \sigma(5^2) = 15 \times 31 = 465\,.$$

All the positive factors of 200 sum to 465.

3. (a) We need to prove that n is an abundant number $\Leftrightarrow \sigma(n) > 2n$.

Proof.

($\Rightarrow$). Let n be an abundant number and $d_1,\ d_2,\ \cdots,\ d_m$ be the proper factors (divisors) of n. By the definition of abundant number we have

$$d_1 + d_2 + \cdots + d_m > n \qquad (*)$$

By the definition of the sigma function we have

$$\sigma(n) = \underbrace{d_1 + d_2 + \cdots + d_m}_{>n \text{ by } (*)} + n > n + n = 2n\,.$$

($\Leftarrow$). Assume $\sigma(n) > 2n$ then

$$\sigma(n) = d_1 + d_2 + \cdots + d_m + n > 2n \;\Rightarrow\; d_1 + d_2 + \cdots + d_m > n\,.$$

Hence n is an abundant number.

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(b) Very similar to proof of part (a) but we use the deficient number property:

$$d_1 + d_2 + \cdots + d_m < n\,.$$

(c) *Proof.*

We have already shown n is a perfect number $\Rightarrow \sigma(n) = 2n$.

Just need to show that if $\sigma(n) = 2n \Rightarrow n$ is a perfect number.

By the definition of the sigma function we have

$$\sigma(n) = (d_1 + d_2 + \cdots + d_m) + n = 2n$$

where d_j are the proper divisors of n. Transposing this equation we have

$$d_1 + d_2 + \cdots + d_m = 2n - n = n$$

This $d_1 + d_2 + \cdots + d_m = n$ implies that n is perfect number.

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Characterising the numbers given in question 2:

15 is deficient because $\sigma(15) = 24 < 2 \times 15 = 30$.

77 is also deficient because $\sigma(77) = 96 < 2 \times 77$.

Since $\sigma(171) = 260$ so 171 is deficient because $260 < 2 \times 171$.

200 is abundant because $\sigma(200) = 465 > 2 \times 200$.

4. We need to find $\sigma(500)$.

The prime decomposition of $500 = 100 \times 5 = 2^2 \times 5^2 \times 5 = 2^2 \times 5^3$.

Since the sigma function is multiplicative and $\gcd(2^2,\ 5^3) = 1$ so

$$\sigma(500) = \sigma(2^2 \times 5^3) = \sigma(2^2) \times \sigma(5^3)$$
$$= \left(\frac{2^{2+1} - 1}{2 - 1}\right) \times \left(\frac{5^{3+1} - 1}{5 - 1}\right) = 1092 \qquad \left[\text{Applying } \sigma(p^k) = \frac{p^{k+1} - 1}{p - 1}\right]$$

Hence $\sigma(500) = 1092$. Adding *all* the positive factors of 500 gives 1092 which means 500 is an abundant number.

5. We need to prove that a prime number is a deficient number.

Proof.

Let p be a prime number. By Proposition (4.32):

We have p is a prime number $\Leftrightarrow \sigma(p) = p + 1$.

Hence we have

$$\sigma(p) = p + 1 < 2p.$$

By the previous question part (b) we have p is a deficient number.

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6. Consider the even integer 10. Then

$$\sigma\left(10\right) = 1 + 2 + 5 + 10 = 18 < 2 \times 10$$

Hence 10 is a deficient even number.

7. By examining the list of the first few perfect numbers given in the main text we have 6, 28, 496, 8128 and 33 550 336. Of course the first four perfect numbers do obey the rule

'There is *one* perfect number for any given number of digits'

However, the next perfect number 33 550 336 has 8 digits so there are no perfect numbers with 5, 6 or 7 digits.

8. (a) We need to show that $\sigma\left(2^n\right) = 2^{n+1} - 1$.

Proof.

The factors of 2^n are

$$1,\ 2,\ 2^2,\ \cdots,\ 2^n.$$

From the definition of the sigma function we have

$$\sigma\left(2^n\right) = 1 + 2 + 2^2 + \cdots + 2^n.$$

How do we find the sum on the right-hand-side?

By applying the geometric series formula (4.29):

$$\boxed{a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a\left(1 - r^n\right)}{1 - r}}$$

Hence

$$\begin{aligned}\sigma\left(2^n\right) &= 1 + 2 + 2^2 + \cdots + 2^n \\ &= \frac{1\left(1 - 2^{n+1}\right)}{1 - 2} = 2^{n+1} - 1 \quad \begin{bmatrix}\text{Multiplying numerator} \\ \text{and denominator by } -1\end{bmatrix}\end{aligned}$$

This completes our proof.

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(b) Very similar proof to part (a) with 2 being replaced by the p.

9. We are asked to prove:

Every *even* perfect number N is of the form:

$$N = 2^{p-1}\left(2^p - 1\right) \text{ where } \left(2^p - 1\right) \text{ is prime.}$$

Proof.

Let $N = 2^s m$ where m is odd and $s \geq 1$, be a perfect even number. Since $N = 2^s \times m$ is a perfect number so

$$\sigma\left(N\right) = 2 \times 2^s m = 2^{s+1} m \qquad (\dagger)$$

Since m is odd and 2^s is even so $\gcd\left(2^s, \ m\right) = 1$. Applying the multiplicative property of the sigma function we have

$$\begin{aligned}\sigma\left(N\right) &= \sigma\left(2^s \times m\right)\\ &= \sigma\left(2^s\right) \times \sigma\left(m\right)\\ &= \left(2^{s+1} - 1\right) \times \sigma\left(m\right) \quad (*) \quad \left[\text{By result of question 8(a)}\right]\end{aligned}$$

Now we consider two cases, m is prime and then m is composite.

<u>Case I</u>: m is prime

As m is prime so $\sigma\left(m\right) = m + 1$. Substituting this into (*) yields

$$\begin{aligned}\sigma\left(N\right) &= \left(2^{s+1} - 1\right) \times \sigma\left(m\right)\\ &= \left(2^{s+1} - 1\right) \times \left(m + 1\right)\\ &\underbrace{=}_{\text{By } (\dagger)} 2^{s+1} m\end{aligned}$$

From (†) we have $\sigma\left(N\right) = 2^{s+1} m$ so equating the above line to this gives

$$\begin{aligned}\sigma\left(N\right) = \left(2^{s+1} - 1\right) \times \left(m + 1\right) &= 2^{s+1} m\\ m + 1 &= \frac{2^{s+1}}{2^{s+1} - 1} m\\ 1 &= \left(\frac{2^{s+1}}{2^{s+1} - 1} - 1\right) m\\ 1 &= \left(\frac{1}{2^{s+1} - 1}\right) m \quad \Rightarrow \quad 2^{s+1} - 1 = m\end{aligned}$$

Putting $m = 2^{s+1} - 1$ into $N = 2^s \times m$ gives $N = 2^s \times \left(2^{s+1} - 1\right)$ which is of the requested form $N = 2^{p-1}\left(2^p - 1\right)$. However we still need to show that s is prime.

How?

By question 13 of Exercises 4(c):

If $2^n - 1$ is prime then n is prime.

As we are considering $m = 2^{s+1} - 1$ is prime so $s + 1 = p$ where p is prime and so

$$N = 2^s\left(2^{s+1} - 1\right) = 2^{p-1}\left(2^p - 1\right)$$

Hence if m is prime we have our required result.

<u>Case II</u>: m is composite.

By equating (*) and (†) from above and re-arranging we have

$$\sigma\left(N\right) = \left(2^{s+1} - 1\right)\sigma\left(m\right) = 2^{s+1} m \qquad (**)$$

From this we have $2^{s+1} \,\Big|\, \left(2^{s+1} - 1\right)\sigma\left(m\right)$. We have $\gcd\left(2^{s+1}, \ 2^{s+1} - 1\right) = 1$. *Why?*

Because 2^{s+1}, $2^{s+1}-1$ are consecutive integers.

By applying Euclid's Lemma (1.13):

If $a \mid (b \times c)$ with $\gcd(a, b) = 1$ then $a \mid c$.

to $2^{s+1} \mid (2^{s+1}-1)\sigma(m)$ gives

$$2^{s+1} \mid \sigma(m) \quad \Rightarrow \quad 2^{s+1}k = \sigma(m) \text{ where } k \text{ is a positive integer.}$$

Substituting this $2^{s+1}k = \sigma(m)$ into (**) yields

$$(2^{s+1}-1)\,2^{s+1}k = 2^{s+1}m \quad \Rightarrow \quad (2^{s+1}-1)k = m \quad \Rightarrow \quad 2^{s+1}k = k + m \qquad (***)$$

Since $s \geq 1$ and $k \mid m$ so $1 < k < m$ because m is composite. This means that k is a proper divisor of m and so

$$\sigma(m) = 1 + m + k + \frac{m}{k} + (\text{any other factors of } m) \qquad (\dagger)$$

Substituting $m = (2^{s+1}-1)k$ into (**) gives

$$\sigma(N) = (2^{s+1}-1)\sigma(m) = 2^{s+1}(2^{s+1}-1)k$$
$$\sigma(m) = 2^{s+1}k$$

From (***) we have $2^{s+1}k = k + m$. Putting this into the above line gives

$$\sigma(m) = 2^{s+1}k = m + k \qquad (\dagger\dagger)$$

This implies that k is the sum of the proper divisors of m. Equating ($\dagger$) and ($\dagger\dagger$):

$$\sigma(m) = 1 + m + k + \frac{m}{k} + (\text{any other factors of } m) = m + k$$
$$\Rightarrow \quad 1 + \frac{m}{k} + (\text{any other factors of } m) = 0$$

This last line is impossible so m *cannot* be composite. Hence m is prime.

This completes our proof.

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10. We need to prove:

We have p is a prime number $\Leftrightarrow \sigma(p) = p + 1$.

Proof.

($\Rightarrow$). Let p be prime. Then the only factors of p are 1 and p, therefore

$$\sigma(p) = p + 1$$

($\Leftarrow$). Assume that $\sigma(n) = n + 1$. Suppose n is composite. Then n has factors 1, n, and $d > 0$ say. Therefore we have

$$\sigma(n) = n + 1 + d > n + 1$$

This is impossible because we are assuming $\sigma(n) = n+1$. Hence n *cannot* be composite so it is a prime, that is $n = p$. This completes our proof.

11. We are asked to prove:

Let the prime decomposition of a positive integer n be given by

$$n = p_1^{k_1} \times p_2^{k_2} \times p_3^{k_3} \times \cdots \times p_m^{k_m} \text{ where } p\text{'s are distinct primes.}$$

Then

$$\sigma(n) = \sigma\left(p_1^{k_1} \times p_2^{k_2} \times p_3^{k_3} \times \cdots \times p_m^{k_m}\right) = \sigma\left(p_1^{k_1}\right) \times \sigma\left(p_2^{k_2}\right) \times \sigma\left(p_3^{k_3}\right) \times \cdots \times \sigma\left(p_m^{k_m}\right)$$

How do we prove this?

Use mathematical induction.

Proof.

The result is true for $m = 2$:

$$\sigma(n) = \sigma\left(p_1^{k_1} \times p_2^{k_2}\right) = \sigma\left(p_1^{k_1}\right) \times \sigma\left(p_2^{k_2}\right)$$

Why?

Because by Proposition (4.36):

$$\sigma(n) = \sigma\left(p^k \times q^m\right) = \sigma\left(p^k\right) \times \sigma\left(q^m\right) \text{ where } p \text{ and } q \text{ are distinct primes.}$$

Assume the result is true for $m = s$:

$$\sigma\left(p_1^{k_1} \times p_2^{k_2} \times p_3^{k_3} \times \cdots \times p_s^{k_s}\right) = \sigma\left(p_1^{k_1}\right) \times \sigma\left(p_2^{k_2}\right) \times \sigma\left(p_3^{k_3}\right) \times \cdots \times \sigma\left(p_s^{k_s}\right) \qquad (*)$$

Required to prove this for $m = s+1$:

$$\sigma\left(p_1^{k_1} \times p_2^{k_2} \times \cdots \times p_s^{k_s} \times p_{s+1}^{k_{s+1}}\right) = \sigma\left(p_1^{k_1}\right) \times \sigma\left(p_2^{k_2}\right) \times \cdots \times \sigma\left(p_s^{k_s}\right) \times \sigma\left(p_{s+1}^{k_{s+1}}\right)$$

Consider the left-hand-side of this equation and apply Proposition (4.36):

$$\begin{aligned} \sigma\left(p_1^{k_1} \times p_2^{k_2} \times \cdots \times p_s^{k_s} \times p_{s+1}^{k_{s+1}}\right) &= \sigma\left(p_1^{k_1} \times p_2^{k_2} \times \cdots \times p_s^{k_s}\right) \times \sigma\left(p_{s+1}^{k_{s+1}}\right) \\ &= \underbrace{\sigma\left(p_1^{k_1}\right) \times \sigma\left(p_2^{k_2}\right) \times \cdots \times \sigma\left(p_s^{k_s}\right)}_{\text{By } (*)} \times \sigma\left(p_{s+1}^{k_{s+1}}\right) \end{aligned}$$

Hence by mathematical induction we have our required result.

12. (a) We are asked to show that $\sigma\left(p^3\right) = \left(p^2+1\right)\left(p+1\right)$.

Proof.

The divisors of p^3 are 1, p, p^2 and p^3. Adding these gives

$$\sigma\left(p^3\right) = 1 + p + p^2 + p^3$$

Expanding the right – hand side of the given formula yields

$$\left(p^2 + 1\right)\left(p + 1\right) = p^3 + p^2 + p + 1$$

Hence, we have our result $\sigma\left(p^3\right) = \left(p^2 + 1\right)\left(p + 1\right)$.

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(b) We need to show that $\sigma\left(p^5\right) = \left(p^2 - p + 1\right)\left(p^2 + p + 1\right)\left(p + 1\right)$.

Proof.

Like part (a) we have the factors of p^5 are 1, p, p^2, p^3, p^4 and p^5. Adding these gives

$$\sigma\left(p^5\right) = 1 + p + p^2 + p^3 + p^4 + p^5$$

Expanding

$$\left(p^2 - p + 1\right)\left(p^2 + p + 1\right)\left(p + 1\right) = 1 + p + p^2 + p^3 + p^4 + p^5$$

This completes our proof.

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13. (i) We are asked to show that the last digit of 2^{2k} is either 4 or 6.

Proof.

For the last digit we consider modulo 10, we are required to prove that

$$2^{2k} \equiv 4, 6 \pmod{10}$$

We use mathematical induction:

For $k = 1$ we have $2^2 \equiv 4 \pmod{10}$, so our result holds.

Assume the result is true for $k = m$:

$$2^{2m} \equiv 4, 6 \pmod{10} \qquad (*)$$

Required to prove $2^{2(m+1)} \equiv 4, 6 \pmod{10}$. Expanding the left-hand-side:

$$\begin{aligned} 2^{2(m+1)} &\equiv 2^{2m+2} \\ &\equiv 2^2 \times 2^{2m} \equiv 4 \times 2^{2m} \equiv 4 \times 4,\ 4 \times 6 \equiv 6,\ 4 \pmod{10} \end{aligned}$$

Hence by mathematical induction we have our result; $2^{2k} \equiv 4, 6 \pmod{10}$.

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(ii) We need to prove the following:

For every even perfect number the last digit is either a 6 or 8.

Proof.

Since we are interested in the last digit so consider modulo 10.

By Theorem (4.30):

Every *even* perfect number N is of the form:

$$N = 2^{p-1} \times \left(2^p - 1\right) \text{ where } \left(2^p - 1\right) \text{ is prime.}$$

If $p = 2$ then we have $N = 2 \times \left(2^2 - 1\right) = 6$. The result is true for the even prime 2.

Let p be an odd prime. Then $p - 1$ is even and let $p - 1 = 2k$. Therefore by part (i) we have

$$2^{p-1} \equiv 2^{2k} \equiv 4, 6 \pmod{10}.$$

Consider each case; (I) $2^{p-1} \equiv 4 \pmod{10}$ and (II) $2^{p-1} \equiv 6 \pmod{10}$:

<u>Case (I)</u> We have $2^{p-1} \equiv 4 \pmod{10}$. Multiplying this by 2 gives

$$2 \times 2^{p-1} \equiv 2^p \equiv 2 \times 4 \equiv 8 \pmod{10}.$$

Subtracting 1 from both sides yields

$$2^p - 1 \equiv 8 - 1 \equiv 7 \pmod{10}.$$

Therefore, evaluating the last digit of $N = 2^{p-1}\left(2^p - 1\right)$ with $2^{p-1} \equiv 4 \pmod{10}$:

$$N = 2^{p-1} \times \left(2^p - 1\right) \equiv 7 \times 4 \equiv 8 \pmod{10}.$$

<u>Case (II)</u> We have $2^{p-1} \equiv 6 \pmod{10}$. Multiplying this by 2 gives

$$2 \times 2^{p-1} \equiv 2^p \equiv 2 \times 6 \equiv 2 \pmod{10}.$$

Subtracting 1 from both sides yields

$$2^p - 1 \equiv 2 - 1 \equiv 1 \pmod{10}.$$

Evaluating the last digit of $N = 2^{p-1}\left(2^p - 1\right)$ with $2^{p-1} \equiv 6 \pmod{10}$:

$$N = 2^{p-1} \times \left(2^p - 1\right) \equiv 1 \times 6 \equiv 6 \pmod{10}.$$

Hence in either case we have the last digit of $N = 2^{p-1} \times \left(2^p - 1\right)$ is either 6 or 8.

This completes our proof.

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14. We need to show $2^{p-1}\left(2^p - 1\right) \equiv 1 \pmod 9$ where p is an odd prime.

Proof.

First note that $2 \equiv -1 \pmod 3$. Since we are given that p is an odd prime so $p - 1$ is even and we have

$$2^{p-1} \equiv (-1)^{p-1} \equiv 1 \pmod 3.$$

We can write this as $2^{p-1} = 3k + 1$ where k is an integer. Therefore

$$2^p - 1 = 2\left(2^{p-1}\right) - 1 = 2(3k+1) - 1 = 6k + 1.$$

Hence, we have

$$\begin{aligned} 2^{p-1}\left(2^p - 1\right) &= (3k+1)(6k+1) \\ &= 18k^2 + 9k + 1 = 9\left(2k^2 + k\right) + 1 \end{aligned}$$

Therefore, we have our result $2^{p-1}\left(2^p - 1\right) \equiv 1 \pmod 9$.

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15. Theorem (4.28) says:

> If the Mersenne number $2^p - 1$ is prime then
> $$N = 2^{p-1}\underbrace{\left(2^p - 1\right)}_{\text{prime}} \text{ is a perfect number.}$$

It says that if $N = 2^{p-1}\left(2^p - 1\right)$ then N is a perfect and *not* that if N is perfect then $N = 2^{p-1}\left(2^p - 1\right)$. In symbolic notation

$$N = 2^{p-1}\left(2^p - 1\right) \quad \Rightarrow \quad N \text{ is a perfect number.}$$
$$N \text{ is a perfect number} \quad \not\Rightarrow \quad N = 2^{p-1}\left(2^p - 1\right).$$

16. (a) We need to prove that $m \times N$ where $m > 1$ is an abundant number.

Proof.

We are given that N is a perfect number. Let $d_1, d_2, \cdots, d_k$ be the positive divisors of m and $e_1, e_2, \cdots, e_l$ be the positive divisors of N. Both the d's and e's are divisors of $m \times N$. Since N is a perfect number so

$$e_1 + e_2 + \cdots + e_l = 2N$$

Then

$$d_1 + d_2 + \cdots + d_k + \underbrace{e_1 + e_2 + \cdots + e_l}_{=2N} = d_1 + d_2 + \cdots + d_k + 2N > 2N$$

Then $\sigma(N) \geq d_1 + d_2 + \cdots + d_k + 2N > 2N$. By the result of question 3(a):

> n is an abundant number $\Leftrightarrow \sigma(n) > 2n$.

We conclude that $m \times N$ is an abundant number.

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(b) We need to show that $\dfrac{N}{d}$ where d is a proper divisor of N is a deficient number.

Proof.

Let $d_1,\ d_2,\ \cdots,\ d_k$ be the positive divisors of N. Since N is a perfect number so

$$\sigma\left(N\right) = d_1 + d_2 + \cdots + d_k = 2N\ .$$

We are given that d is a proper divisor of N so $d = d_j$ is one amongst the list $d_1,\ d_2,\ \cdots,\ d_k$. WLOG assume $d = d_1$ then

$$\sigma\left(\frac{N}{d}\right) = d_2 + \cdots + d_k < 2N\ .$$

By the result of question 3(b):

$$\boxed{n \text{ is an deficient number} \Leftrightarrow \sigma\left(n\right) < 2n\ .}$$

We conclude that $m \times N$ is a deficient number.

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