Complete Solutions to Exercises 1.1

1. We need to find $\gcd(60,\ 84)$. The positive divisors of 60 are

$$\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 10,\ 12,\ 15,\ 20,\ 30,\ 60\}$$

Which of these numbers are also divisors of 84?

$$\{1,\ 2,\ 3,\ 4,\ 6,\ 12\}$$

Therefore $\gcd(60,\ 84) = 12$. We need to cut our metal into 12 inch squares.

2. The positive divisors of 57 are $\{1,\ 3,\ 19,\ 57\}$. *Which of these are also divisors of 209?*

$$\{1,\ 19\}$$

Hence $\gcd(57,\ 209) = 19$.

Rewriting the numerator and denominator of $\frac{57}{209}$ in terms of 19:

$$\frac{57}{209} = \frac{3 \times \cancel{19}}{11 \times \cancel{19}} = \frac{3}{11} \qquad \text{[Cancelling]}$$

3. The divisors of 65 are $\{1,\ 5,\ 13,\ 65\}$. *Which of these numbers also go into 1001?*

$$\{1,\ 13\}$$

Thus $\gcd(65,\ 1001) = 13$. Dividing each number, 65 and 1001, by 13 gives 5 and 77 respectively. Therefore, the ratio $65 : 1001$ can be written in its simplest form as $5 : 77$.

4. The greatest common divisor is 100Hz.

5. (a) The positive divisors of 10 are $\{1, 2, 5, 10\}$ therefore ${}^{2}(10) = 4$.
(b) The positive divisors of 100 are $\{1, 2, 4, 5, 10, 20, 25, 50, 100\}$ therefore ${}^{2}(100) = 9$.
(c) The positive divisors of 120 are

$$\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12,\ 15,\ 20,\ 24,\ 30,\ 40,\ 60,\ 120\}$$

therefore $\tau(120) = 16$.
(d) The number 101 is a prime number so $\tau(101) = 2$.

6. Don't let the negative numbers put you off because the gcd is positive. This means we can ignore the negative sign.

(a) To find the $\gcd(-12,\ 34)$, it is easier to look at the divisors of 34 because 12 has too many divisors. The divisors of 34 are

$$\{1,\ 2,\ 17,\ 34\}$$

Only 1 and 2 go into 12 so $\gcd(-12,\ 34) = 2$.

(b) The positive divisors of -36 are

$$\{1,\ 2,\ 3,\ 4,\ 6,\ 9,\ 12,\ 18,\ 36\}$$

The largest number in this set which also goes into 60 or -60 is 12 therefore

$$\gcd(-36,\ -60) = 12$$

(c) The divisors of 60 are

$$\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 10,\ 12,\ 15,\ 20,\ 30,\ 60\}$$

The positive divisors of -72 are

$$\{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 9,\ 12,\ 18,\ 24,\ 36,\ 72\}$$

The greatest common divisor is 12.

(d) We need to find $\gcd(1001,\ 182)$. Life is a lot simpler if we look at the divisors of the smaller number 182:

$$\{1,\ 2,\ 7,\ 13,\ 14,\ 26,\ 91,\ 182\}$$

Which of these numbers also go into 1001?

None of the even numbers because 1001 is odd. Only the following from the above set go into 1001:

$$\{1,\ 7,\ 13,\ 91\}$$

Therefore $\gcd(1001,\ 182) = 91$.

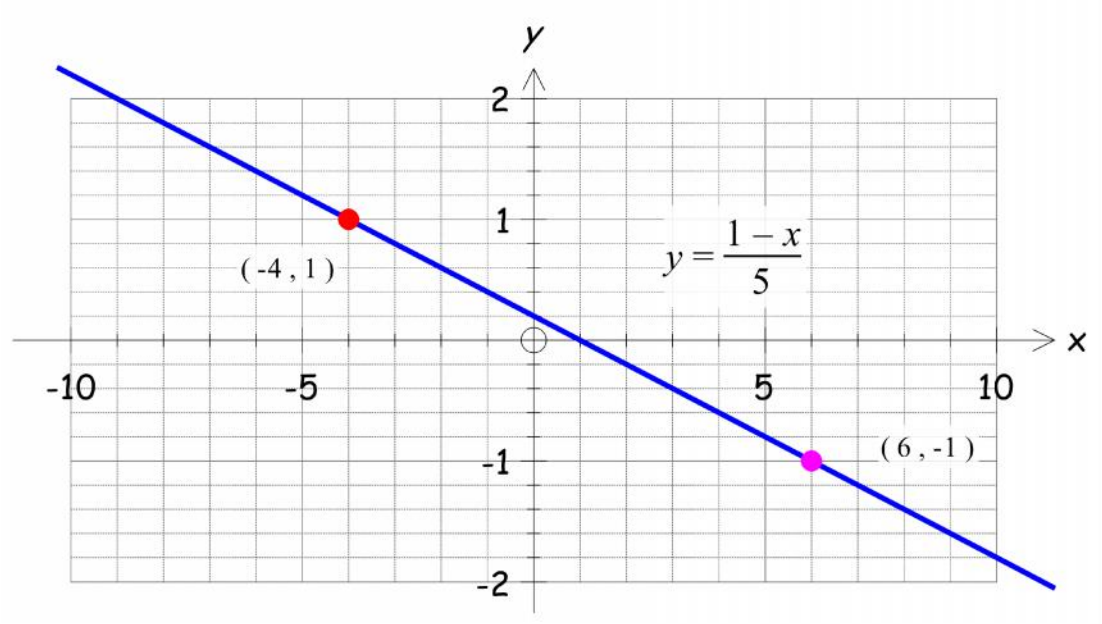
7. The equation $24x + 120y = \gcd(24,\ 120)$ is a straight line. Since 24 goes into 120 so the $\gcd(24,\ 120) = 24$. We have the equation

$$24x + 120y = 24$$

Dividing this equation by 24 gives

$$x + 5y = 1 \quad \Rightarrow \quad y = \frac{1}{5}(1 - x)$$

This straight line has gradient $-\dfrac{1}{5}$ and $y-$intercept $\dfrac{1}{5}$:



The graph shows two integer solutions $x = -4, y = 1$ and $x = 6, \ y = -1$.

8. We are given the equation $56x + 60y = \gcd(56, \ \ 60)$. *What is* $\gcd(56, \ \ 60)$ *equal to?*

   The divisors of 56 are $\{1, 2, 4, 7, 8, 14, 28, 56\}$. *Which of these numbers also go into 60?*

   $\{1, \ 2, \ 4\}$. Hence $\gcd(56, \ \ 60) = 4$. We need to solve

   $$56x + 60y = 4$$

   You could obtain the solution $x = -1, \ y = 1$ by inspection.

9. We need to find $\gcd(66, \ \ 165, \ \ 253)$. The divisors of 66 are

   $$\{1, 2, 3, 6, 11, 22, 33, 66\}$$

   Which of these numbers go into 165?

   $$\{1, 3, 11, 33\}$$

   Which of the numbers in the last set go into 253*?*

   $$\{1, \ 11\}$$

   Therefore $\gcd(66, \ \ 165, \ \ 253) = 11$.

10. Required to find $\gcd(a, \ \ a^2)$. Clearly a goes into a^2. *Is the gcd equal to a?*

No because gcd is a positive integer and a maybe negative. Therefore, we take the modulus and $\gcd\left(a,\quad a^2\right)=\left|a\right|$.

11. Remember a^2-b^2 is difference of two squares:

$$a^2-b^2=\left(a-b\right)\left(a+b\right)$$

The factor $a+b$ is common between both the integers so

$$\gcd\left(a+b,\quad a^2-b^2\right)=\gcd\left(a+b,\quad\left(a+b\right)\left(a-b\right)\right)=\left|a+b\right|$$

12. The given statement, $a\mid b$ and $c\mid d$ implies $\left(a+c\right)\mid\left(b+d\right)$, is false because

$$2\mid 4 \text{ and } 3\mid 9 \quad\text{but } 5\nmid 13.$$

13. (a) We are given $a\mid 0$ which means there is an integer m such that $am=0$.

Let m equal zero then a can be any integer.

(b) We are given $a\mid 2$ so a is a factor of 2 which means we have $a=\pm1,\ \pm2$.

14. *Proof.*

We have $a\mid b \iff ax=b \iff axc=bc \iff ac\mid bc$.

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15. We are given $a\mid\left(b+c\right)$ and $a\mid b$. We need to deduce that $a\mid c$.

Proof.

Since $a\mid\left(b+c\right)$ so there is an integer x such that

$$ax=b+c \qquad (*)$$

From $a\mid b$ implies there is an integer y such that

$$ay=b$$

Substituting this $b=ay$ into (*) yields

$$ax=ay+c \Rightarrow ax-ay=c \Rightarrow a\left(x-y\right)=c \Rightarrow a\mid c$$

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16. Since $b^2-c^2=\left(b+c\right)\left(b-c\right)$ and we are given $a\mid b,\ a\mid c$ which implies $a\mid\left(b+c\right)$ therefore $a\mid\left(b+c\right)\left(b-c\right) \Rightarrow a\mid\left(b^2-c^2\right)$.

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17. A counter example of this is:

$$22 \mid (11 \times 12) \text{ but } 22 \nmid 11 \text{ or } 22 \nmid 12$$

We will show in the next question that if $d \mid (n_1 \times n_2)$ where $d > 1$ and $\gcd(n_1, \ n_2) = 1$ then only one of the following is true $d \mid n_1$ or $d \mid n_2$.

18. (a) We are asked to prove that two consecutive integers are relatively prime.

Proof.

Let n and $n+1$ be two consecutive integers. We need to show that gcd of n and $n+1$ is 1. We prove this by contradiction.

Suppose $g = \gcd(n, \ n+1) > 1$ then

$$g \mid n \text{ and } g \mid (n+1)$$

Applying the Linear Combination Theorem (1.3):

If $a \mid b$ and $a \mid c$ then $a \mid (bx + cy)$

To $g \mid (n+1)$ and $g \mid n$ with $x = 1$ and $y = -1$ yields

$$g \mid [(n+1) - n] \quad \Rightarrow \quad g \mid 1$$

This result $g \mid 1$ is impossible because $g > 1$. Hence our supposition is wrong and two consecutive integers are relatively prime.

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(b) Similar proof to part (a).

19. We have to prove:

If $m \mid n$ where $m < n$ (m is a proper divisor) then $m \leq \frac{n}{2}$.

Proof.

Suppose $m > \frac{n}{2}$. We are given that $m \mid n$ and $m < n$ so there exists an integer $k > 1$ such that $mk = n$. As $k > 1$ so $k \geq 2$ which implies

$$mk > \frac{n}{2} 2 = n$$

This is a contradiction because $mk = n$. Hence our supposition must be wrong so $m \leq \frac{n}{2}$.

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20. We need to show that if n is an odd integer then $\gcd\left(n+1,\quad n^2+1\right)=2$.

Proof.

Let $n=2m+1$ be odd, then $n+1=2\left(m+1\right)$ and

$$\begin{aligned} n^2+1 &= \left(2m+1\right)^2+1 \\ &= 4m^2+4m+2 = 2\left(2m^2+2m+1\right) = 2\underbrace{\left(2m\left(m+1\right)+1\right)}_{\text{odd}} \end{aligned}$$

Thus we have

$$\gcd\left(n+1,\quad n^2+1\right)=\gcd\left(2\left(m+1\right),\quad 2\left(2m\left(m+1\right)+1\right)\right) \qquad (*)$$

The integers $m+1$ and $2\left(m^2+m\right)+1$ have *no* factor in common. That is

$$\gcd\left(m+1,\quad 2m\left(m+1\right)+1\right)=1$$

Why?

Suppose $\gcd\left(m+1,\quad 2\left(m^2+m\right)+1\right)=g>1$ then

$$g\,\Big|\,\left(m+1\right) \quad\text{and}\quad g\,\Big|\,\left[2\left(m^2+m\right)+1\right]$$

Applying the Linear Combination Theorem (1.3):

If $a\,\big|\,b$ and $a\,\big|\,c$ then $a\,\big|\,\left(bx+cy\right)$ for any integers x and y.

To the above $g\,\Big|\,\left(m+1\right)$ and $g\,\Big|\,\left[2\left(m^2+m\right)+1\right]$ with $b=-2m$ and $c=1$ gives

$$g\,\Big|\,\left[-2m\left(m+1\right)+2m\left(m+1\right)+1\right] \;\Rightarrow\; g\,\Big|\,1$$

This $g\,\Big|\,1$ is a contradiction because in our supposition we had $g>1$. Hence $g=1$. By (*) we have

$$\gcd\left(n+1,\quad n^2+1\right)=\gcd\left(2\left(m+1\right),\quad 2\left(2m\left(m+1\right)+1\right)\right)=2$$

This completes our proof.

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21. We are asked to prove:

If $d\,\Big|\,\left(n_1\times n_2\right)$ where $d>1$ and $\gcd\left(n_1,\quad n_2\right)=1$ then only one of the following is true; $d\,\Big|\,n_1$ or $d\,\Big|\,n_2$.

Proof.

We need to dismiss two cases;

Case 1) $d \nmid n_1$ and $d \nmid n_2$. If this is the case then there are *no* integers x and y such that $dx = n_1$ and $dy = n_2$. This implies that $n_1 \times n_2$ *cannot* be a multiple of *d*. However, we are given that $d \mid (n_1 \times n_2)$ which implies that $n_1 \times n_2$ is a multiple of *d*. Hence, we cannot have $d \nmid n_1$ and $d \nmid n_2$.

Case II) Suppose $d > 1$ divides both the n's, that is $d \mid n_1$ and $d \mid n_2$. Then $\gcd(n_1,\ n_2) \geq d > 1$ which is impossible because $\gcd(n_1,\ n_2) = 1$.

Since we cannot have either of these cases, so we must have one of the following; $d \mid n_1$ or $d \mid n_2$. This completes our proof.

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22. *Proof.*

Suppose $g = \gcd(-a,\ -b) > \gcd(a,\ b) = d$. That is $g > d$.

Since $g = \gcd(-a,\ -b)$ so

$$g \mid (-a) \text{ and } g \mid (-b)$$

This means there are integers x and y such that

$$gx = -a \text{ and } gy = -b$$

Multiplying both equations by -1 gives

$$g(-x) = a \text{ and } g(-y) = b$$

Therefore, g is a common divisor of a and b. However $\gcd(a,\ b) = d$ so by:

> Definition (1.4).
> (ii) If $c \mid a$ and $c \mid b$ then $c \leq d$. [d is the largest of the common divisors]

We have $g \leq d$. This is a contradiction because in the above supposition we have $g > d$. This supposition $g > d$ must be wrong.

Producing a similar argument to the above we can show that if we suppose $g < d$ then we have a contradiction.

We have $g \not> d$ and $g \not< d$ therefore $g = d$.

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23. We use proof by induction to prove this result.

Proof.

The result is true for $n = 1$ because we are given $a \mid b_1$ so $a \mid b_1 x_1$.

Assume the result is true for $n = k$ which implies there is an integer m such that

$$am = b_1x_1 + b_2x_2 + \cdots + b_kx_k \qquad (*)$$

Required to prove the result for $n = k+1$, that is we need to show

$$a\left(\text{integer}\right) = b_1x_1 + b_2x_2 + \cdots + b_kx_k + b_{k+1}x_{k+1} \qquad (\dagger)$$

Consider the right-hand-side of (†):

$$\underbrace{b_1x_1 + b_2x_2 + \cdots + b_kx_k}_{=am \text{ by } (*)} + b_{k+1}x_{k+1} = am + b_{k+1}x_{k+1}$$

We are also given that $a \mid b_{k+1}$ which means there is an integer q such that:

$$aq = b_{k+1}$$

Putting this into the above equation gives

$$\underbrace{b_1x_1 + b_2x_2 + \cdots + b_kx_k}_{=am \text{ by } (*)} + b_{k+1}x_{k+1} = am + aq = a\left(m+q\right)$$

We have shown (†) therefore by mathematical induction we have our result;

$$a \mid \left(b_1x_1 + b_2x_2 + \cdots + b_nx_n\right)$$

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