Complete Solutions to Exercises 8.3

1. (a) We need to find the least positive solution of $x^2 - 3y^2 = 1$ because we are given that $N = 3 = 2^2 - 1$.

Transposing the given equation $x^2 - 3y^2 = 1$ we have

$$x^2 = 1 + 3y^2 \quad \Rightarrow \quad x = \sqrt{1 + 3y^2}$$

For $y = 1$ we have $x = \sqrt{1+3} = 2$. Hence a solution is $x = 2$ and $y = 1$.

(b) Similarly we need to find the least positive solution of

$$x^2 - 7y^2 = 1$$

Transposing this to make x the subject gives

$$x^2 = 1 + 7y^2 \quad \Rightarrow \quad x = \sqrt{1 + 7y^2}$$

Creating a table of values:

y	1	2	3	4
$x = \sqrt{1+7y^2}$	$\sqrt{8}$	$\sqrt{29}$	$\sqrt{64} = 8$	*

Since for $y = 3$ we have $x = \sqrt{64} = 8$ so we don't need to evaluate any further values of x. Our least solution is $x = 8,\ y = 3$.

(c) Note that $N = 8$ is one less than a square number 9. Therefore the equation $x^2 - 8y^2 = 1$ which is transposed to $x = \sqrt{1 + 8y^2}$ will give us integer values for $y = 1$. Hence $x = \sqrt{1+8} = 3$. Our solution is $x = 3, y = 1$.

(d) We are asked to find the least positive solution of

$$x^2 - 11y^2 = 1$$

Transposing this we have $x = \sqrt{1 + 11y^2}$. Creating a table of values:

y	1	2	3	4
$x = \sqrt{1+11y^2}$	$\sqrt{12}$	$\sqrt{45}$	$\sqrt{100} = 10$	*

Again by sheer coincidence $y = 3$ works. Hence our least positive solution to $x^2 - 11y^2 = 1$ is $x = 10,\ y = 3$.

(e) Transposing the given equation $x^2 - 10y^2 = 1$ to making x the subject:

$$x = \sqrt{1 + 10y^2}$$

Creating a table of values:

y	1	2	3	4	5	6
$x = \sqrt{1+10y^2}$	$\sqrt{11}$	$\sqrt{41}$	$\sqrt{91}$	$\sqrt{161}$	$\sqrt{251}$	$\sqrt{361} = 19$

Therefore with $y = 6$ we have a square number 361 whose square root is 19.

Our least (fundamental) positive solution to $x^2 - 10y^2 = 1$ is $x = 19,\ \ y = 6$.

(f) Similarly for $N = 12$ we have to solve $x^2 - 12y^2 = 1$. Transposing gives

$$x = \sqrt{1 + 12y^2}$$

Substituting $y = 2$ into this $x = \sqrt{1 + 12y^2}$ yields

$$x = \sqrt{1 + \left(12 \times 2^2\right)} = \sqrt{49} = 7$$

Our least positive solution is $x = 7$, $y = 2$.

2. First we need to check that $r = 649 + 180\sqrt{13}$ produces a solution to

$$x^2 - 13y^2 = 1$$

Substituting $x = 649$ and $y = 180$ into $x^2 - 13y^2$ yields

$$649^2 - \left(13 \times 180^2\right) = 1$$

Another solution is given by r^2:

$$\begin{aligned} r^2 = \left(649 + 180\sqrt{13}\right)^2 &= 649^2 + \left(2 \times 649 \times 180\sqrt{13}\right) + \left(180^2 \times 13\right) \\ &= 421\ 201 + 233\ 640\sqrt{13} + 421\ 200 \\ &= 842\ 401 + 233\ 640\sqrt{13} \end{aligned}$$

Checking that this $r^2 = 842\ 401 + 233\ 640\sqrt{13}$ does produce a solution by substituting $x = 842\ 401$ and $y = 233\ 640$ into $x^2 - 13y^2$ gives

$$x^2 - 13y^2 = 842401^2 - \left(13 \times 233640^2\right) = 1$$

Hence $x = 842401$ and $y = 233640$ is a solution of $x^2 - 13y^2 = 1$.

3. (i) Substituting $x = 577$ and $y = 408$ into the given equation:

$$x^2 - 2y^2 = 577^2 - \left(2 \times 408^2\right) = 1$$

(ii) From part (i) we have $x^2 - 2y^2 = 1$. Transposing this gives

$$x^2 - 1 = 2y^2 \quad \Leftrightarrow \quad \frac{x^2 - 1}{y^2} = 2 \quad \Leftrightarrow \quad \sqrt{\frac{x^2 - 1}{y^2}} = \sqrt{2}$$

For large x we have $\sqrt{x^2 - 1} \simeq \sqrt{x}$ therefore

$$\sqrt{2} \simeq \sqrt{\frac{x^2}{y^2}} = \frac{x}{y} \quad \text{(for large } x\text{)}$$

Substituting $x = 577$ and $y = 408$ into this yields

$$\sqrt{2} \simeq \frac{x}{y} = \frac{577}{408}$$

The discrepancy is given by

$$\frac{577}{408}-\sqrt{2}=0.000\ 002\ 124=2.124\times 10^{-6}$$

This is an error of about 2 parts in a million.

(iii) We are asked to show $\frac{x}{y}=\sqrt{2+\frac{1}{y^2}}$ gives an approximation for $\sqrt{2}$.

Rearranging the given equation $x^2-2y^2=1$ we have

$$x^2-2y^2=1 \;\Leftrightarrow\; x^2=1+2y^2 \;\Leftrightarrow\; \frac{x^2}{y^2}=\frac{1}{y^2}+2$$

Taking the square root of both sides gives

$$\frac{x}{y}=\sqrt{2+\frac{1}{y^2}}$$

To get good approximation we need large y because then $\frac{1}{y^2}\simeq 0$.

(b) Very similar to part (iii) solution.

4. (i) Verify that $r^2=17+12\sqrt{2}$. Using this to determine r^3 gives

$$\begin{aligned} r^3=r^2\times r &= \left(17+12\sqrt{2}\right)\left(3+2\sqrt{2}\right)\\ &= 51+34\sqrt{2}+36\sqrt{2}+\left(24\times 2\right)\\ &= 99+70\sqrt{2}\end{aligned}$$

We check that $r^3=99+70\sqrt{2}$ does indeed provide a solution to $x^2-2y^2=1$ by substituting $x=99,\ y=70$ into x^2-2y^2:

$$x^2-2y^2=99^2-\left(2\times 70^2\right)=1$$

Now we use this irrational number $r^3=99+70\sqrt{2}$ to find r^4:

$$\begin{aligned} r^4=r^3\times r &= \left(99+70\sqrt{2}\right)\left(3+2\sqrt{2}\right)\\ &= 297+198\sqrt{2}+210\sqrt{2}+\left(140\times 2\right)\\ &= 577+408\sqrt{2}\end{aligned}$$

Again, checking this solution by substituting $x=577,\ y=408$ into x^2-2y^2 yields

$$x^2-2y^2=577^2-\left(2\times 408^2\right)=1$$

Hence $r^4 = 577 + 408\sqrt{2}$ provides a solution $x = 577,\ y = 408$ to $x^2 - 2y^2 = 1$.

(This solution was given in question 3.)

(ii) Approximating $\sqrt{2}$ by using $r = 3 + 2\sqrt{2}$, $r^2 = 17 + 12\sqrt{2}$, $r^3 = 99 + 70\sqrt{2}$ and $r^4 = 577 + 408\sqrt{2}$ gives $\frac{3}{2}, \frac{17}{12}, \frac{99}{70}$ and $\frac{577}{408}$ respectively. The discrepancy is found by subtracting $\sqrt{2}$ from these rational approximations:

$$\frac{3}{2} - \sqrt{2} = 0.0858 \text{ (3sf)}$$

$$\frac{17}{12} - \sqrt{2} = 0.00245 \text{ (3sf)}$$

$$\frac{99}{70} - \sqrt{2} = 0.0000722 \text{ (3sf)}$$

$$\frac{577}{408} - \sqrt{2} = 0.00000212 \text{ (3sf)} \quad \left[\text{Found in question 3}\right]$$

(iii) We have the equation $x^2 - 2y^2 = 1$. Transposing this gives

$$x^2 - 1 = 2y^2 \quad \text{implies} \quad \frac{x^2 - 1}{y^2} = 2$$

Taking the square root of both sides of this $\frac{x^2 - 1}{y^2} = 2$ gives

$$\sqrt{\frac{x^2 - 1}{y^2}} = \frac{\sqrt{x^2 - 1}}{y} = \sqrt{2}$$

For large x we have $\sqrt{x^2 - 1} \approx \sqrt{x^2} = x$.

Hence the ratio $\frac{x}{y}$ of positive solutions of $x^2 - 2y^2 = 1$ gives rational approximations to $\sqrt{2}$.

5. We are asked to show that the least positive solution to $x^2 - Ny^2 = 1$ where $N = n^2 - 1$ is given by $x = n,\ y = 1$.

Proof.

Substituting $N = n^2 - 1$ into $x^2 - Ny^2 = 1$ gives

$$x^2 - Ny^2 = x^2 - \left(n^2 - 1\right)y^2 = 1$$

Transposing this $x^2 - \left(n^2 - 1\right)y^2 = 1$ to make x the subject gives

$$x=\sqrt{1+\left(n^2-1\right)y^2}$$

Substituting $y=1$ into this $x=\sqrt{1+\left(n^2-1\right)y^2}$ yields

$$x=\sqrt{1+\left(n^2-1\right)}=\sqrt{n^2}=n$$

Hence a solution to $x^2-Ny^2=1$ where $N=n^2-1$ is $x=n,\ y=1$.

We also need to show $x=n,\ y=1$ is the *least* positive solution. *How?*

By Proposition (8.20):

Let $r=a+b\sqrt{N}$ and $s=c+d\sqrt{N}$ both produce positive solutions of Pell's equation $x^2-Ny^2=1$. Then $r<s \Leftrightarrow a<c$.

In our case let $r=n+\sqrt{N}$ and $s=c+d\sqrt{N}$. Suppose $s<r$. Then by this Proposition (8.20) we have $c<n$.

Making y the subject of the given equation $x^2-Ny^2=1$:

$$x^2-1=Ny^2 \quad\Leftrightarrow\quad \frac{x^2-1}{N}=y^2 \quad\Leftrightarrow\quad \sqrt{\frac{x^2-1}{N}}=y$$

Substituting $1\le x=c<n$ and $N=n^2-1$ into $y=\sqrt{\frac{x^2-1}{N}}$ gives the inequality

$$y=\sqrt{\frac{x^2-1}{N}}=\sqrt{\frac{c^2-1}{n^2-1}}<\sqrt{\frac{\cancel{n^2-1}}{\cancel{n^2-1}}}=1 \qquad \left[\text{Cancelling}\right]$$

We have $y<1$. Since we are interested in non-trivial positive solutions so

$$y \not< 1 \ [y \text{ cannot be less than } 1].$$

This is a contradiction therefore our supposition $s<r$ must be wrong and so $r=n+\sqrt{N}$ produces the least positive solution to $x^2-Ny^2=1$ where $N=n^2-1$. Hence $x=n,\ y=1$ is the least positive solution.

■

We use this result $x=n,\ y=1$ to find the least positive solution of the given N.

(a) We need to find the least positive solution of $x^2-15y^2=1$. Since $15=4^2-1$ so by the above result we have $x=4,\ y=1$.

(b) Similarly the least positive solution of $x^2 - 24y^2 = 1$ is $x = 5,\ y = 1$ because $24 = 5^2 - 1$.

(c) The least positive solution of $x^2 - 35y^2 = 1$ is $x = 6,\ y = 1$ because $35 = 6^2 - 1$.

(d) For $x^2 - 48y^2 = 1$ the least positive solution is $x = 7,\ y = 1$ because $48 = 7^2 - 1$.

6. (i) We need to check $r = 17 + 12\sqrt{2}$ yields a solution to $x^2 - 2y^2 = 1$.
Substituting $x = 17$, $y = 12$ into $x^2 - 2y^2$ gives

$$x^2 - 2y^2 = 17^2 - \left(2 \times 12^2\right) = 1$$

Therefore $r = 17 + 12\sqrt{2}$ produces a solution to $x^2 - 2y^2 = 1$.

(ii) We need to solve $u^2 - 8v^2 = 1$. We can rewrite this as

$$u^2 - 8v^2 = u^2 - 2\left(2v\right)^2 = 1$$

Let $u = x$ and $v = \dfrac{y}{2}$ then

$$u^2 - 8v^2 = u^2 - 2\left(2v\right)^2 = x^2 - 2\left(\cancel{2} \times \frac{y}{\cancel{2}}\right)^2 = x^2 - 2y^2 = 1$$

We solved this $x^2 - 2y^2 = 1$ in part (i) with solution $x = 17$, $y = 12$. Therefore $u = x = 17$ and $v = \dfrac{y}{2} = \dfrac{12}{2} = 6$ is a solution to $u^2 - 8v^2 = 1$.

(iii) This time we need to solve $u^2 - 32v^2 = 1$. Similarly

$$u^2 - 32v^2 = u^2 - 2\left(4v\right)^2 = 1$$

Let $u = x$ and $v = \dfrac{y}{4}$ then

$$u^2 - 32v^2 \underbrace{=}_{\text{because } 32 = 2\times 4^2} u^2 - 2\left(4v\right)^2 = x^2 - 2\left(\cancel{4} \times \frac{y}{\cancel{4}}\right)^2 = x^2 - 2y^2 = 1$$

So we have $u = x = 17$ and $v = \dfrac{y}{4} = \dfrac{12}{4} = 3$ is a solution to $u^2 - 32v^2 = 1$.

(iv) We are asked to find the condition on m in $u^2 - 2\left(mv\right)^2 = 1$ to have integer solutions.

From solution to part (i) we have $r = 17 + 12\sqrt{2}$ yields a solution to $x^2 - 2y^2 = 1$.

So let $u = x$ and $v = \dfrac{y}{m}$ then

$$u^2 - 2\left(mv\right)^2 = x^2 - 2\left(\cancel{m}\frac{y}{\cancel{m}}\right)^2 = x^2 - 2y^2 = 1$$

From above we have $u = x = 17$ and $v = \dfrac{y}{m} = \dfrac{12}{m}$. Hence m must be a factor of 12 because for Pell's equation we are interested in integer solutions only.

7. Trialling integer values of $y = 1,\ 2,\ 3,\ \cdots, 7, 8, \cdots$ we find that we have integer x when $y = 7$ because

$$x = \sqrt{1 + 47y^2} = \sqrt{1 + \left(47 \times 7^2\right)} = 48$$

This $x = 48$, $y = 7$ is our least solution because smaller values of y do not give integer values of $x = \sqrt{1 + 47y^2}$ (try it).

Let $r = 48 + 7\sqrt{47}$. Two more solutions are given by

$$\begin{aligned} r^2 &= \left(48 + 7\sqrt{47}\right)^2 \\ &= 48^2 + \left(2 \times 48 \times 7\sqrt{47}\right) + \left(7^2 \times 47\right) \\ &= 4607 + 672\sqrt{47} \end{aligned}$$

Another solution is $x = 4607,\ y = 672$. The third solution is found by

$$\begin{aligned} r^3 &= r^2 \times r \\ &= \left(4607 + 672\sqrt{47}\right)\left(48 + 7\sqrt{47}\right) \\ &= \left(4607 \times 48\right) + \left(4607 \times 7\sqrt{47}\right) + \left(48 \times 672\sqrt{47}\right) + \left(672 \times 7 \times 47\right) \\ &= 442\ 224 + 64\ 505\sqrt{47} \end{aligned}$$

Therefore a third solution is $x = 442\ 224,\ y = 64\ 505$.

8. We are asked to show that if N is a square number then $x^2 - Ny^2 = 1$ only has trivial solutions; $x = 1,\ y = 0$ or $x = -1,\ y = 0$.

Proof.

Let $N = n^2 > 0$. So the given equation is $x^2 - n^2y^2 = 1$.

Using the difference of two squares on $x^2 - n^2y^2$ gives

$$x^2 - n^2y^2 = \left(x - ny\right)\left(x + ny\right) = 1$$

The only way that $\left(x-ny\right)\left(x+ny\right)=1$ is if

$$x-ny=x+ny=1 \text{ or } x-ny=x+ny=-1$$

Solving the first case $x-ny=x+ny=1$:

$$x-x-ny-ny=0 \;\Rightarrow\; -2ny=0 \;\Rightarrow\; y=0$$

Substituting $y=0$ into the above $x-ny=x+ny=1$ gives $x=1$.

Similarly solving the other case $x-ny=x+ny=-1$ yields $x=-1,\ y=0$.

This completes our proof.

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9. We need to check that $r=55+12\sqrt{21}$ produces a solution of $x^2-21y^2=1$.

Substituting $x=55$ and $y=12$ into x^2-21y^2:

$$x^2-21y^2=55^2-\left(21\times 12^2\right)=1$$

Hence $r=55+12\sqrt{21}$ produces a solution to $x^2-21y^2=1$. This $x=55$ and $y=12$ is the least positive solution to $x^2-21y^2=1$ because for $y<12$ the value of $x=\sqrt{1+21y^2}$ will *not* be an integer. You can trial all the integers from 1 to 11 to check this. So by Proposition (8.20):

> Let $r=a+b\sqrt{N}$ and $s=c+d\sqrt{N}$ both produce positive solutions of Pell's equation $x^2-Ny^2=1$. Then $r<s \;\Leftrightarrow\; a<c$.

The irrational number $r=55+12\sqrt{21}$ produces the least positive solution.

We can find two other positive solutions by evaluating r^2 and r^3:

$$\begin{aligned} r^2=\left(55+12\sqrt{21}\right)^2 &= 55^2+\left(2\times 55\times 12\sqrt{21}\right)+\left(12\sqrt{21}\right)^2 \\ &= 3025+1320\sqrt{21}+\left(144\times 21\right) \\ &= 6049+1320\sqrt{21} \end{aligned}$$

Before evaluating r^3 we need to check that $r^2=6049+1320\sqrt{21}$ does produce a solution because we want to use this answer to find r^3.

Substituting $x=6049,\ y=1320$ into x^2-21y^2:

$$x^2-21y^2=6049^2-\left(21\times 1320^2\right)=1$$

Hence $r^2=6049+1320\sqrt{21}$ does indeed produce a solution to $x^2-21y^2=1$.

Finding r^3:

$$
\begin{aligned}
r^3 = r^2 \times r &= \left(6049 + 1320\sqrt{21}\right)\left(55 + 12\sqrt{21}\right) \\
&= \left(6049 \times 55\right) + \left(6049 \times 12\sqrt{21}\right) + \left(1320 \times 55\sqrt{21}\right) + \left(1320\sqrt{21} \times 12\sqrt{21}\right) \\
&= 332\ 695 + 725\ 88\sqrt{21} + 72\ 600\sqrt{21} + 332\ 640 \\
&= 665\ 335 + 145\ 188\sqrt{21}
\end{aligned}
$$

Checking that $r^3 = 665\ 335 + 145\ 188\sqrt{21}$ does indeed produce a solution to $x^2 - 21y^2 = 1$ by substituting $x = 665\ 335,\ \ y = 145\ 188$ into $x^2 - 21y^2$:

$$x^2 - 21y^2 = 665335^2 - \left(21 \times 145188^2\right) = 1.$$

Our two additional solutions are

$$x = 6049,\ \ y = 1320 \ \text{ and } \ x = 665\ 335,\ \ y = 145\ 188.$$

10. We are asked to prove that:

For integers a, b, c, d and N we have the identity:

$$\left(a^2 - Nb^2\right)\left(c^2 - Nd^2\right) = \left(ac + Nbd\right)^2 - N\left(ad + bc\right)^2$$

Proof.

Expanding the right - hand side gives

$$
\begin{aligned}
\left(ac + Nbd\right)^2 - N\left(ad + bc\right)^2 &= a^2c^2 + 2acbdN + N^2b^2d^2 - N\left(a^2d^2 + 2adbc + b^2c^2\right) \\
&= a^2c^2 + 2acbdN + N^2b^2d^2 - a^2d^2N - 2adbcN - b^2c^2N \\
&\underbrace{=}_{\substack{\text{Because} \\ 2acbdN - 2adbcN = 0}} a^2c^2 + N^2b^2d^2 - a^2d^2N - b^2c^2N \qquad (*)
\end{aligned}
$$

Expanding the left - hand side gives

$$\left(a^2 - Nb^2\right)\left(c^2 - Nd^2\right) = a^2c^2 - Na^2d^2 - Nb^2c^2 + N^2b^2d^2 \qquad (\dagger)$$

Since (*) and (†) are identical so our identity holds.

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11. We are asked to prove:

If $r = a + b\sqrt{N}$ produces a positive solution of Pell's equation $x^2 - Ny^2 = 1$ then so does $r^n = \left(a + b\sqrt{N}\right)^n = \alpha + \beta\sqrt{N}$ where $n = 1,\ 2,\ 3, \cdots$.

Proof.

If we expand $\left(a + b\sqrt{N}\right)^n$ by the binomial theorem then

$$r^n = \left(a + b\sqrt{N}\right)^n = \alpha + \beta\sqrt{N}$$

Now we use mathematical induction on n.

Clearly the result is true for $n = 1$ because

$$r^1 = r = a + b\sqrt{N}$$

And we are given that $r = a + b\sqrt{N}$ produces a solution to $x^2 - Ny^2 = 1$.

Assume the result is true for $n = k$, that is

$$r^k \text{ produces a solution to } x^2 - Ny^2 = 1$$

Required to prove the result for $n = k + 1$:

$$r^{k+1} = r^k \times r$$

By the induction hypothesis we have r^k produces a solution to $x^2 - Ny^2 = 1$ and we are given that r produces a solution to this equation. By Proposition (8.18):

> If the irrational numbers r and s produces a solution of Pell's equation $x^2 - Ny^2 = 1$ then so does the *product* $r \times s$.

The product $r^{k+1} = r^k \times r$ produces a solution to $x^2 - Ny^2 = 1$.

By mathematical induction we conclude that r^n where $n > 0$ produces a solution to $x^2 - Ny^2 = 1$.

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12. We need to prove the following:

> If $r = a + b\sqrt{N}$ produces a solution of Pell's equation $x^2 - Ny^2 = 1$ then so does $r^n = \left(a + b\sqrt{N}\right)^n$ where n is any integer.

Looks like the result of the previous question but this time we have extended n to be any integer.

Proof.

We divide the proof into three cases; Case I: $n = 0$, Case II: $n > 0$ and Case III: $n < 0$

<u>Case I</u>: Consider $n = 0$.

With $n = 0$ we have

$$r^0 = 1 = a + b\sqrt{N} \;\Rightarrow\; a = 1,\;\; N = 0$$

We have the trivial solution $x = 1$ and y is any integer.

<u>Case II</u>: Consider $n > 0$.

We proved this part in the previous question.

<u>Case III</u>: Consider $n < 0$.

This time we apply mathematical induction to negative integers.

For $n = -1$ we have r^{-1}. By Proposition (8.17):

> If the irrational number r produces a solution of Pell's equation $x^2 - Ny^2 = 1$ then so does the *reciprocal* $\frac{1}{r} = r^{-1}$.

We have r^{-1} produces a solution to $x^2 - Ny^2 = 1$.

Assume the result is correct for $n = -k$:

$$r^{-k} \text{ produces a solution to } x^2 - Ny^2 = 1$$

Consider $n = -\left(k+1\right)$. We have

$$r^{-n} = r^{-\left(k+1\right)} = r^{-k} \times r^{-1}$$

By the induction hypothesis r^{-k} produces a solution to $x^2 - Ny^2 = 1$ and from the reciprocal rule we have r^{-1} produces a solution to $x^2 - Ny^2 = 1$. Therefore by the product proposition we have $r^{-\left(k+1\right)} = r^{-k} \times r^{-1}$ produces a solution to $x^2 - Ny^2 = 1$.

Hence by mathematical induction we conclude that r^n where $n < 0$ produces a solution to $x^2 - Ny^2 = 1$.

This completes our proof.

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13. We are asked to prove:

> Let $r = a + b\sqrt{N}$ and $s = c + d\sqrt{N}$ both produce positive solutions of Pell's equation $x^2 - Ny^2 = 1$. Then $r < s \iff a < c$.

We need to prove this both ways because we are given $\Leftrightarrow$ in the proposition.

Proof.

We are given that both r and s are positive solutions so $r > 0$ and $s > 0$.

$\left(\Leftrightarrow\right)$. We can prove implication and being implied by together because the inequalities below go both ways, $\Rightarrow$ and $\Leftarrow$.

Since $r < s$ so

$$r = a + b\sqrt{N} < c + d\sqrt{N} = s$$

Suppose $a \geq c$ (we use proof by contradiction).

We are given that $r = a + b\sqrt{N}$ and $s = c + d\sqrt{N}$ both produce solutions to $x^2 - Ny^2 = 1$. Therefore

$$a^2 - Nb^2 = c^2 - Nd^2 = 1$$

Re-arranging this $a^2 - Nb^2 = c^2 - Nd^2$ we have

$$a^2 - c^2 = Nb^2 - Nd^2 = N\left(b^2 - d^2\right) \qquad (*)$$

In our supposition we have $a \geq c$ which implies

$$a^2 \geq c^2 \quad \Leftrightarrow \quad a^2 - c^2 \geq 0$$

Putting this inequality into (*) yields

$$N\left(b^2 - d^2\right) \geq 0 \underbrace{\Leftrightarrow}_{\text{because } N>0} b^2 - d^2 \geq 0 \Leftrightarrow b \geq d$$

From the last inequality $b \geq d$ we have

$$b\sqrt{N} \geq d\sqrt{N} \quad \Leftrightarrow \quad a + b\sqrt{N} \geq a + d\sqrt{N} \underbrace{\Leftrightarrow}_{\text{because } a\geq c} a + b\sqrt{N} \geq c + d\sqrt{N}$$

From this last inequality $a + b\sqrt{N} \geq c + d\sqrt{N}$ we have $r \geq s$. *Why?*

Because we are given $r = a + b\sqrt{N}$ and $s = c + d\sqrt{N}$. This is a contradiction because we are assuming $r < s$. Hence $a \geq c$ is wrong, so $a < c$.

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14. We need to prove that *all* the positive solutions of $x^2 - 6y^2 = 1$ are produced by

$$r^n = \left(5 + 2\sqrt{6}\right)^n \text{ where } n \text{ is a natural number.}$$

Proof.

We know by Example 12 that $r = 5 + 2\sqrt{6}$ produces a solution to $x^2 - 6y^2 = 1$.

This $r = 5 + 2\sqrt{6}$ produces the least positive solution of $x^2 - 6y^2 = 1$.

By Proposition (8.19):

> If $r = a + b\sqrt{N}$ produces a solution of Pell's equation $x^2 - Ny^2 = 1$ then so does $r^n = \left(a + b\sqrt{N}\right)^n$ where $n = 1, 2, 3, \cdots$.

We have

$$r^n = \left(5 + 2\sqrt{6}\right)^n \text{ also produces a solution of } x^2 - 6y^2 = 1$$

Let x_0 and y_0 be an arbitrary positive solution to $x^2 - 6y^2 = 1$. Then

$$x_0^{\ 2} - 6y_0^{\ 2} = 1$$

Writing this as the difference of two squares

$$x_0^{\ 2} - 6y_0^{\ 2} = \left(x_0 - \sqrt{6}y_0\right)\left(x_0 + \sqrt{6}y_0\right) = 1$$

We have $x_0 + y_0\sqrt{6}$ produces a positive solution to $x^2 - 6y^2 = 1$. Required to prove

$$x_0 + y_0\sqrt{6} = \left(5 + 2\sqrt{6}\right)^n \text{ for some } n$$

Suppose for every n we have

$$x_0 + y_0\sqrt{6} \neq \left(5+2\sqrt{6}\right)^n \qquad (*)$$

There must be an integer k such that

$$\left(5+2\sqrt{6}\right)^k < x_0 + y_0\sqrt{6} < \left(5+2\sqrt{6}\right)^{k+1}$$

Dividing this inequality through by the positive number $\left(5+2\sqrt{6}\right)^k$ gives

$$1 < \frac{x_0 + y_0\sqrt{6}}{\left(5+2\sqrt{6}\right)^k} < 5+2\sqrt{6}$$

Recall that both $x_0 + y_0\sqrt{6}$ and $\left(5+2\sqrt{6}\right)^k$ produce solutions to $x^2 - 6y^2 = 1$.

By the reciprocal and product propositions we have

$$\frac{x_0 + y_0\sqrt{6}}{\left(5+2\sqrt{6}\right)^k} \text{ produce solutions to } x^2 - 6y^2 = 1$$

Therefore $1 < \dfrac{x_0 + y_0\sqrt{6}}{\left(5+2\sqrt{6}\right)^k} < 5+2\sqrt{6}$ gives us a contradiction. *Why?*

Because $5+2\sqrt{6}$ produces the *least positive* solution but we have $\dfrac{x_0 + y_0\sqrt{6}}{\left(5+2\sqrt{6}\right)^k}$

which satisfies $1 < \dfrac{x_0 + y_0\sqrt{6}}{\left(5+2\sqrt{6}\right)^k} < 5+2\sqrt{6}$ also producing a solution.

Our supposition that $x_0 + y_0\sqrt{6} \neq \left(5+2\sqrt{6}\right)^n$ must be wrong so

$$x_0 + y_0\sqrt{6} = \left(5+2\sqrt{6}\right)^n \text{ for some } n.$$

Hence *all* the solutions of $x^2 - 6y^2 = 1$ must be given by $r^n = \left(5+2\sqrt{6}\right)^n$.

■

15. This is very similar to proof of the previous question.