Complete Solutions to Exercises 1.4

1. (a) We need to solve $2x+3y=5$. The $\gcd(2,\ \ 3)=1$ and 1 divides 5 so we have integer solutions. An integer solution to

$$2x'+3y'=1 \quad \text{is} \quad x'=-1 \quad \text{and} \quad y'=1$$

An integer solution to $2x+3y=5$ is $x_0=-1\times 5=-5$ and $y_0=1\times 5=5$.

The general solution is given by Proposition (1.18):

$$\boxed{x=x_0+\left(\frac{b}{g}\right)t \text{ and } y=y_0-\left(\frac{a}{g}\right)t \quad \text{where } t \text{ is any integer.}}$$

Substituting $a=2,\ b=3,\ g=1,\ x_0=-5$ and $y_0=5$ into this gives

$$x=x_0+\left(\frac{b}{g}\right)t=-5+\left(\frac{3}{1}\right)t=-5+3t\ ,\ y=y_0-\left(\frac{a}{g}\right)t=5-\left(\frac{2}{1}\right)t=5-2t$$

Our general solution is $x=3t-5$ and $y=5-2t$ where t is any integer.

Note that your general solution may not look like this because you may have a different initial solution such as $x_0=1,\ y_0=1$. Then your general solution would be $x=3t+1$ and $y=1-2t$. We can find particular solutions to the given Diophantine equation by substituting various t values into this general solution $x=3t+1$ and $y=1-2t$.

(b) Since $\gcd(3,\ \ 6)=3$ and 3 divides 9 so the equation $3x+6y=9$ has integer solutions.

An initial solution of $3x+6y=9$ is $x_0=3,\ y_0=0$. Using Proposition (1.18) with $a=3,\ b=6,\ g=3,\ x_0=3$ and $y_0=0$:

$$x=x_0+\left(\frac{b}{g}\right)t=3+\left(\frac{6}{3}\right)t=3+2t\ ,$$

$$y=y_0-\left(\frac{a}{g}\right)t=0-\left(\frac{3}{3}\right)t=-t$$

The general solution to $3x+6y=9$ is $x=3+2t$, $y=-t$.

(c) We are required to solve $15x-20y=10$. The $\gcd(15,\ \ -20)=5$. Solving the easier equation $15x'-20y'=5$ by using Euclidean algorithm:

$$20=15+5 \quad \text{implies} \quad 20-15=5$$

$20-15=5$ can be rewritten as $15(-1)-20(-1)=5$ which gives $x'=-1,\ y'=-1$ as a solution to $15x'-20y'=5$. Hence a solution to $15x-20y=10$ is

$$x_0 = -1 \times 2 = -2 \quad \text{and} \quad y_0 = -1 \times 2 = -2$$

Substituting $a = 15$, $b = -20$, $g = 5$, $x_0 = -2$ and $y_0 = -2$ into the formula of (1.18):

$$x = x_0 + \left(\frac{b}{g}\right)t = -2 + \left(\frac{-20}{5}\right)t = -2 - 4t \quad \text{and} \quad y = y_0 - \left(\frac{a}{g}\right)t = -2 - \left(\frac{15}{5}\right)t = -2 - 3t$$

The solution is $x = -2 - 4t$ and $y = -2 - 3t$.

2. (a) The $\gcd(2, \ 4) = 2$ and 2 does not divide 1 so $2x + 4y = 1$ has no solutions.

(b) We must solve $48x + 56y = 32$. Applying the Euclidean algorithm to 48 and 56 gives

$$56 = 48 + 8 \qquad (*)$$
$$48 = 6(8) + 0$$

The $\gcd(48, \ 56) = 8$ and $8 \mid 32$ so the equation $48x + 56y = 32$ has solutions.

Let us use the above results to solve $48x' + 56y' = 8$. By (*) we have

$$48(-1) + 56(1) = 8 \text{ which gives } x' = -1, \ y' = 1$$

Since $8 \times 4 = 32$ so a solution to $48x + 56y = 32$ is

$$x_0 = -1 \times 4 = -4, \ y_0 = 1 \times 4 = 4.$$

Substituting $a = 48$, $b = 56$, $g = 8$, $x_0 = -4$ and $y_0 = 4$ into the formula of (1.18):

$$x = x_0 + \left(\frac{b}{g}\right)t = -4 + \left(\frac{56}{8}\right)t = -4 + 7t,$$
$$y = y_0 - \left(\frac{a}{g}\right)t = 4 - \left(\frac{48}{8}\right)t = 4 - 6t$$

Thus, the general solution is $x = 7t - 4$, $y = 4 - 6t$.

(c) In order to solve $54x + 180y = -72$ we first need to find the gcd of 54 and 180. Applying the Euclidean algorithm, we have

$$180 = 3(54) + 18 \qquad (\dagger)$$
$$54 = 3(18) + 0$$

Therefore $g = \gcd(180, \ 54) = 18$ and $18 \mid (-72)$ so the equation

$$54x + 180y = -72$$

has integer solutions.

Since we have already used the Euclidean algorithm, we can solve the equation

$$54x' + 180y' = 18$$

By (†) we have $180-3(54)=18$ which gives a solution $x'=-3,\ y'=1$ to

$$54x'+180y'=18$$

However, we need to solve the given equation $54x+180y=-72$. *How?*

Since $-72=-4\times 18$ so we multiply $x'=-3,\ y'=1$ by -4 to give a solution

$$x_0=-3\times(-4)=12 \quad \text{and} \quad y_0=1\times(-4)=-4$$

Hence a solution to $54x+180y=-72$ is $x_0=12$ and $y_0=-4$. The general solution is given by the formula of (1.18) with

$a=54,\ b=180,\ g=18,\ x_0=12$ and $y_0=-4$:

$$x=x_0+\left(\frac{b}{g}\right)t=12+\left(\frac{180}{18}\right)t=12+10t,$$

$$y=y_0-\left(\frac{a}{g}\right)t=-4-\left(\frac{54}{18}\right)t=-4-3t$$

The general solution to $54x+180y=-72$ is $x=12+10t$ and $y=-4-3t$.

3. (a) We need to solve $101x+600y=1001$. First we determine the gcd of 101 and 600 by using the Euclidean algorithm:

$$\begin{aligned}
600 &= 5(101)+95\\
101 &= 95+6\\
95 &= 15(6)+5\\
6 &= 1(5)+\boxed{1} \quad \leftarrow \text{Last non-zero remainder}\\
5 &= 5(1)+0
\end{aligned}$$

Hence $\gcd(101,\ 600)=1$.

By using the above Euclidean algorithm, we can solve the easier equation

$$101x'+600y'=1$$

Reversing the above steps gives

$$\begin{aligned}
1 &= 6-5\\
&= 6-(95-15(6))\\
&= 16(6)-95\\
&= 16(101-95)-95\\
&= 16(101)-17(95)\\
&= 16(101)-17(600-5(101))\\
&= 101(101)-17(600)
\end{aligned}$$

We have $101(101)-17(600)=1$. Therefore a solution to $101x'+600y'=1$ is $x'=101,\ y'=-17$. *How do we find a solution to the given equation*

$$101x + 600y = 1001\ ?$$

Since $1001 = 1001 \times 1$ so we multiply $x' = 101,\ y' = -17$ by 1001:

$$x_0 = 101 \times 1001 = 101\,101,\ \ y_0 = -17 \times 1001 = -17\ 017$$

The general solution of $101x + 600y = 1001$ is found using the formula of (1.18) with $a = 101,\ b = 600,\ g = 1,\ \ x_0 = 101\,101,\ y_0 = -17\ 017$:

$$x = x_0 + \left(\frac{b}{g}\right)t = 101\,101 + 600t\,,$$

$$y = y_0 - \left(\frac{a}{g}\right)t = -17\ 017 - 101t$$

The general solution to $101x + 600y = 1001$ is

$$x = 101\,101 + 600t\,,\ y = -17\ 017 - 101t$$

(b) Using the hint in the question, we have the following from the solution of question 2(d) Exercise 1(c):

The gcd of 181 and 232 is 1 $(g = 1)$ and the solution to

$$181x' + 232y' = 1 \text{ is } x' = -91,\ \ y' = 71$$

But we need to solve the equation $181x + 232y = -100$. Therefore multiplying $x' = -91,\ y' = 71$ by -100 gives a solution

$$x_0 = -91 \times (-100) = 9100,\ \ y_0 = 71 \times (-100) = -7100$$

The general solution to $181x + 232y = -100$ is given by using formula (1.18) with $a = 181,\ b = 232,\ g = 1,\ x_0 = 9100$ and $y_0 = -7100$:

$$x = x_0 + \left(\frac{b}{g}\right)t = 9100 + 232t\,,\ y = y_0 - \left(\frac{a}{g}\right)t = -7100 - 181t$$

Therefore $x = 9100 + 232t,\ y = -7100 - 181t$ is the general solution of

$$181x + 232y = -100$$

4. Let x and y be the number of bars and rolls respectively. We have

$$2x + 3y = 20$$

The gcd of 2 and 3 is 1 and 1 divides 20 so there are integers x and y such that $2x + 3y = 20$. Solving the easier equation:

$$2x' + 3y' = 1$$

By guesswork we have a solution to this $x' = -1,\ \ y' = 1$. *How do we solve*

$$2x + 3y = 20\ ?$$

We can find a solution by multiplying $x' = -1,\ \ y' = 1$ by 20 which gives

$$x_0 = -20, \quad y_0 = 20$$

The general solution is found by using $x = x_0 + \left(\frac{b}{g}\right)t$, $y = y_0 - \left(\frac{a}{g}\right)t$ with

$a = 2,\ b = 3,\ g = 1,\ x_0 = -20$ and $y_0 = 20$:

$$x = -20 + 3t \text{ and } y = 20 - 2t\,.$$

Remember x and y represent the number of bars and rolls respectively. So, these are positive or zero. We need to find integer values t such that

$$x = -20 + 3t \geq 0 \text{ and } \quad y = 20 - 2t \geq 0$$

Solving the first inequality

$$-20 + 3t \geq 0 \quad \Leftrightarrow \quad 3t \geq 20 \quad \Leftrightarrow \quad t \geq \frac{20}{3} = 6.67\,.$$

Solving the second inequality

$$20 - 2t \geq 0 \;\; \Leftrightarrow \;\; 20 \geq 2t \;\; \Leftrightarrow \;\; 10 \geq t \text{ or } t \leq 10$$

Hence t is an integer between 6.67 and 10. This means t can take values 7, 8, 9 and 10.

Substituting $t = 7, 8, 9$ and 10 into $x = -20 + 3t$ and $y = 20 - 2t$ gives

$$x = -20 + 3t = -20 + 3(7) = 1 \text{ and } y = 20 - 2t = 20 - 2(7) = 6$$
$$x = -20 + 3t = -20 + 3(8) = 4 \text{ and } y = 20 - 2t = 20 - 2(8) = 4$$
$$x = -20 + 3t = -20 + 3(9) = 7 \text{ and } y = 20 - 2t = 20 - 2(9) = 2$$
$$x = -20 + 3t = -20 + 3(10) = 10 \text{ and } y = 20 - 2t = 20 - 2(10) = 0$$

The question says you must purchase one of each so we *cannot* have the last solution $x = 10$ and $y = 0$. Hence we have the solutions

$$x = 1 \text{ and } y = 6\,,\ x = 4 \text{ and } y = 4 \text{ or } x = 7 \text{ and } y = 2$$

The list of solutions is

1 bar and 6 rolls or 4 bars and 4 rolls or 7 bars and 2 rolls

5. Let x and y be the number of times we fill the 4 and 5 gallon containers. The equation for this is given by

$$4x + 5y = 3$$

We can solve $4x' + 5y' = 1$ because $\gcd(4,\ 5) = 1$ and $1 \mid 3$. A solution to this

$$4x' + 5y' = 1$$

is $x' = -1$ and $y' = 1$. *What is a solution to* $4x + 5y = 3$ *?*

Multiplying $x' = -1$ and $y' = 1$ by 3 gives

$$x_0 = -3 \text{ and } y_0 = 3$$

which is a solution to $4x + 5y = 3$. This means we can empty out the 4 gallon container 3 times and fill in the 5 gallon container 3 times.

(The general solution is found using the formula $x = x_0 + \left(\frac{b}{g}\right)t$, $y = y_0 - \left(\frac{a}{g}\right)t$ with $a = 4$, $b = 5$, $g = 1$, $x_0 = -3$ and $y_0 = 3$:

$$x = x_0 + \left(\frac{b}{g}\right)t = -3 + 5t, \; y = y_0 - \left(\frac{a}{g}\right)t = 3 - 4t$$

In this case t can take on any integer values.)

6. We need to solve the following:
Using just 5p (£0.05) and 10p (£0.10) pieces, how many of each do you need in order to pay a parking meter of £3.10?
Let x and y be the number of 5p and 10p coins respectively. Then we must solve the equation

$$0.05x + 0.1y = 3.1$$

Multiplying this by 100 gives the equivalent equation

$$5x + 10y = 310$$

The $\gcd(5, \; 10) = 5$ and $5 \mid 310$ so $5x + 10y = 310$ has integer solutions. One solution is $x_0 = 0$, $y_0 = 31$. Using the formula $x = x_0 + \left(\frac{b}{g}\right)t$, $y = y_0 - \left(\frac{a}{g}\right)t$ with $a = 5$, $b = 10$, $g = 5$, $x_0 = 0$ and $y_0 = 31$ gives:

$$x = x_0 + \left(\frac{b}{g}\right)t = 0 + 2t \text{ and } y = y_0 - \left(\frac{a}{g}\right)t = 31 - t$$

The number of 5p and 10p coins cannot be negative so we need to solve the inequalities:

$$x = 2t \geq 0 \text{ and } 31 - t \geq 0$$

From the second inequality $31 - t \geq 0$ we have $t \leq 31$. From the first inequality $x = 2t \geq 0$ we have t is positive or zero. Therefore, t is an integer between 0 and 31 (inclusive). With $t = 0$ we have the above solution $x_0 = 0$, $y_0 = 31$.

Developing a table of values:

t	$x = 2t$ (Number of 5p coins)	$y = 31 - t$ (Number of 10p coins)
0	0	31
1	2	30

2	4	29
$\vdots$	$\vdots$	$\vdots$
31	62	0

7. Let x and y be the number of first- and second-class stamps respectively. The equation is given by

$$0.6x + 0.5y = 50$$

Multiplying this by 10 gives

$$6x + 5y = 500$$

The gcd of 5 and 6 is 1 and 1 divides 500. Solving the easier equation

$$6x' + 5y' = 1$$

gives $x' = 1$ and $y' = -1$. Multiplying this by 500 gives the solution

$$x_0 = 500 \text{ and } y_0 = -500$$

to $6x + 5y = 500$. Using the formula $x = x_0 + \left(\frac{b}{g}\right)t$, $y = y_0 - \left(\frac{a}{g}\right)t$ with $a = 6,\ b = 5,\ g = 1,\ x_0 = 500$ and $y_0 = -500$:

$$x = x_0 + \left(\frac{b}{g}\right)t = 500 + 5t \text{ and } y = y_0 - \left(\frac{a}{g}\right)t = -500 - 6t$$

In this case x and y need to be positive or zero because x and y represent the number of stamps. We need to solve both the inequalities

$$x = 500 + 5t \geq 0 \text{ and } y = -500 - 6t \geq 0\,.$$

From $x = 500 + 5t \geq 0$ we have $t \geq -100$, solving the other inequality

$$-500 - 6t \geq 0 \quad \Leftrightarrow \quad -500 \geq 6t \quad \Leftrightarrow \quad -\frac{500}{6} = -83.3 \geq t \quad \text{or} \quad t \leq -83.3$$

This means that t is an integer between -100 and -83.3. Hence

$$t = -84,\ -85,\ -86,\ \cdots, -100$$

The general solution is $x = 500 + 5t \geq 0$ and $y = -500 - 6t \geq 0$ with the above values of t. Solutions can be found by substituting these t values:

$$x = 500 + 5t = 500 + 5(-84) = 80 \text{ and } y = -500 - 6t = -500 - 6(-84) = 4$$
$$x = 500 + 5t = 500 + 5(-85) = 75 \text{ and } y = -500 - 6t = -500 - 6(-85) = 10$$
$$x = 500 + 5t = 500 + 5(-86) = 70 \text{ and } y = -500 - 6t = -500 - 6(-86) = 16$$
$$\vdots$$
$$x = 500 + 5t = 500 + 5(-100) = 0 \text{ and } y = -500 - 6t = -500 - 6(-100) = 100$$

Recall x is the number of first class stamps and y is the number of second class stamps.

8. Formulating the equation

$$0.24x + 0.14y = 5$$

where x and y represent the number of hotdogs and buns respectively. Multiplying this by 100 gives the equivalent equation

$$24x + 14y = 500$$

The $\gcd(24,\ 14) = 2$ and $2 \mid 500$ so $24x + 14y = 500$ has integer solutions. First let us solve the easier equation $24x + 14y = 2$. Using the Euclidean algorithm we have

$$\begin{aligned} 24 &= 1(14) + 10 \\ 14 &= 1(10) + 4 \\ 10 &= 2(4) + 2 \end{aligned}$$

Working backwards:

$$\begin{aligned} 2 &= 10 - 2(4) \\ &= 10 - 2(14 - 10) \\ &= 3(10) - 2(14) \\ &= 3(24 - 14) - 2(14) = 3(24) - 5(14) \end{aligned}$$

A solution to $24x + 14y = 2$ is $x' = 3$ and $y' = -5$. However, we need to solve $24x + 14y = 500$. Since $2 \times 250 = 500$ so a solution to this is

$$x_0 = 3 \times 250 = 750 \text{ and } y_0 = -5 \times 250 = -1250.$$

Using the formula $x = x_0 + \left(\frac{b}{g}\right)t$, $y = y_0 - \left(\frac{a}{g}\right)t$ with

$a = 24,\ b = 14,\ g = 2,\ x_0 = 750$ and $y_0 = -1250$ gives:

$$x = x_0 + \left(\frac{b}{g}\right)t = 750 + 7t \text{ and } y = y_0 - \left(\frac{a}{g}\right)t = -1250 - 12t$$

The number of buns and hotdogs cannot be negative so we need to solve:

$$x = 750 + 7t \geq 0 \text{ and } y = -1250 - 12t \geq 0$$

Solving these inequalities gives

$$750 + 7t \geq 0 \quad \Leftrightarrow \quad 7t \geq -750 \quad \Leftrightarrow \quad t \geq -\frac{750}{7} = -107.14$$

$$-1250 - 12t \geq 0 \ \Leftrightarrow \ -12t \geq 1250 \ \Leftrightarrow \ 12t \leq -1250 \ \Leftrightarrow \ t \leq -\frac{1250}{12} = -104.16$$

Hence $-107.14 \le t \le -104.16$. Since t can only be an integer so it can only take the values $-107,\ -106$ and -105. Substituting each of these t values gives:

t	$x = 750 + 7t$ (hotdogs)	$y = -1250 - 12t$ (buns)
-107	$x = 750 + 7(-107) = 1$	$y = -1250 - 12t = -1250 - 12(-107) = 34$
-106	$x = 750 + 7(-106) = 8$	$y = -1250 - 12t = -1250 - 12(-106) = 22$
-105	$x = 750 + 7(-105) = 15$	$y = -1250 - 12t = -1250 - 12(-105) = 10$

List of combinations of the number of hotdogs and buns is 1 and 34 or 8 and 22 or 15 and 10 respectively.

9. Let x be the number of £10 notes and y be the number of £20 notes. Then the equation we need to solve is

$$10x + 20y = 100$$

The $g = \gcd(10,\ \ 20) = 10$ and $10 \mid 100$ so we have integer solutions to this equation.

By guesswork a solution to this is $x_0 = 10,\ y_0 = 0$. Applying the formula (1.18) with $x = x_0 + \left(\frac{b}{g}\right)t$, $y = y_0 - \left(\frac{a}{g}\right)t$ with $a = 10,\ b = 20,\ g = 10,\ x_0 = 10$ and $y_0 = 0$ gives:

$$x = x_0 + \left(\frac{b}{g}\right)t = 10 + \left(\frac{20}{10}\right)t = 10 + 2t$$

$$y = y_0 - \left(\frac{a}{g}\right)t = 0 - \left(\frac{10}{10}\right)t = -t$$

Since x and y represent the number of notes so they must be positive or zero. Thus, we have

$$x = 10 + 2t \ge 0 \quad \Leftrightarrow \quad 2t \ge -10 \quad \Leftrightarrow \quad t \ge -5$$

And

$$y = -t \ge 0 \quad \Leftrightarrow \ t \le 0$$

Therefore $-5 \le t \le 0$ so $t = -5,\ -4,\ -3,\ -2,\ -1,\ 0$. Putting these in a table gives

t	$x = 10 + 2t$	$y = -t$
-5	$x = 10 + (2 \times (-5)) = 0$	5
-4	$x = 10 + (2 \times (-4)) = 2$	4
-3	$x = 10 + (2 \times (-3)) = 4$	3

-2	$x = 10 + \left(2 \times \left(-2\right)\right) = 6$	2
-1	$x = 10 + \left(2 \times \left(-1\right)\right) = 8$	1
0	$x = 10 + \left(2 \times 0\right) = 10$	0

Recall that x is the number £10 notes and y is the number of £20 notes. We could have any of the following combinations:
5 lots of £20 and no £10, 4 lots of £20 and 2 lots of £10, 3 lots of £20 and 4 lots of £10, 2 lots of £20 and 6 lots of £10, 1 of £20 and 8 lots of £10, 10 lots of £10 and no £20.

10. We use Proposition (1.18):

> Let $\gcd\left(a,\ b\right) = g$. If $g \mid c$ and x_0, y_0 are particular solutions of the equation
> $$ax + by = c$$
> Then *all* the other solutions of this equation are given by
> $$x = x_0 + \left(\frac{b}{g}\right)t \text{ and } y = y_0 - \left(\frac{a}{g}\right)t$$
> where t is any integer.

Proof.

In our case we have $g = 1$ and $1 \mid c$ so applying this Proposition (1.18) with $g = 1$ gives

$$x = x_0 + \left(\frac{b}{1}\right)t = x_0 + bt \text{ and } y = y_0 - \left(\frac{a}{1}\right)t = y_0 - at$$

Hence, we have our required solution.

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11. *Proof.*

Using Proposition (1.17):

> Let $\gcd\left(a,\ b\right) = g$. The equation $ax + by = c$ has integer solutions $\Leftrightarrow$ $g \mid c$.

Applying this proposition to $ax + by = 1$ with $g = \gcd\left(a,\ b\right) = 1$ gives

$$ax + by = 1 \text{ has integer solutions } \Leftrightarrow 1 \mid 1.$$

This completes our proof.

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12. We are required to prove:

    The equation $ax+by=1$ has *no* solutions provided $\gcd(a,\ b)>1$.

    We give two proofs.

    First Proof

    Take the contrapositive of the result of the previous question.

    Second Proof.

    Proof by contradiction.

    Suppose $ax+by=1$ has solutions x_0 and y_0. We have $ax_0+by_0=1$.

    Let $g=\gcd(a,\ b)>1$. Then $g \mid a$ and $g \mid b$. By Proposition (1.3):

If $c \mid a$ and $c \mid b$ then $c \mid (ax+by)$ for any integers x and y.

    Applying this proposition to $g \mid a$ and $g \mid b$ gives

    $$g \mid (ax_0+by_0)$$

    Since $ax_0+by_0=1$ so $g \mid 1$. However, Theorem (1.2) (a) says:

For integers a we have $a \mid 1 \Leftrightarrow a=\pm 1$

    We have $g=\gcd(a,\ b)>1$ so it cannot equal ± 1. We have a contradiction so our supposition $ax+by=1$ has solutions must be wrong, hence it has no solutions.

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13. The given equation $45x+81y=1$ has no solutions because

    $$\gcd(45,\ 81)=9>1$$

    This follows from the result of the previous question.

14. We need to prove the following:

    Let $\gcd(a,\ b)=1$ and a positive integer k divide c. Let x_0, y_0 be particular solutions of the equation

    $$akx+bky=c$$

    Then *all* the other solutions of this equation are given by

    $$x=x_0+\left(\frac{b}{k}\right)t \text{ and } y=y_0-\left(\frac{a}{k}\right)t$$

    where t is any integer.

Proof.

Again, we use Proposition (1.18) to prove this result.

Rewriting the given equation

$$akx + bky = (ak)x + (bk)y = c$$

What is the gcd of ak and bk?

Using Proposition (1.11):

Let $\gcd(a, \; b) = g$. For any positive integer m

$$\gcd(ma, \; mb) = mg$$

We have

$$\gcd(ka, \; kb) = k \times 1 = k \qquad \left[\text{Because } \gcd(a, \; b) = 1\right]$$

We are also given that $k \mid c$ so we have integer solutions to $(ak)x + (bk)y = c$.

Applying Proposition (1.18):

Let $\gcd(a, \; b) = g$. If $g \mid c$ and x_0, y_0 are particular solutions of the equation

$$ax + by = c$$

Then *all* the other solutions of this equation are given by

$$x = x_0 + \left(\frac{b}{g}\right)t \text{ and } y = y_0 - \left(\frac{a}{g}\right)t$$

where t is any integer.

With $\gcd(ka, \; kb) = k$ we have all the solutions to $akx + bky = c$ given by

$$x = x_0 + \left(\frac{b}{k}\right)t \text{ and } y = y_0 - \left(\frac{a}{k}\right)t$$

This completes our proof.

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15. We need to prove the following:

The general solution of $ax + may = na$ is given by

$$x = x_0 + mt \text{ and } y = y_0 - t.$$

Proof.

Since we are given that $a \neq 0$ we can multiply $ax + may = na$ by $\dfrac{1}{a}$ to give

$$x + my = n$$

The gcd of 1 and m is 1 and 1 divides n so this equation has integer solutions. Let x_0 and y_0 be particular solutions of this equation $x + my = n$. Applying Corollary

(1.19):

> Corollary (1.19)
> Let $\gcd\left(a,\quad b\right)=1$ and x_0, y_0 be particular solutions of the equation
> $$ax+by=c$$
> Then *all* the other solutions of this equation are given by
> $$x=x_0+bt \text{ and } y=y_0-at$$

To $x+my=n$ with $a=1,\ b=m$ gives

$$x=x_0+mt \text{ and } y=y_0-t.$$

This is our required result.

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16. (a) The given statement:

If $d\mid a$ and $d\mid b$ then the equation $ax+by=c$ has solutions is false.

For example consider the equation $8x+16y=5$. Then $8\mid 8$ and $8\mid 16$. The $\gcd\left(8,\quad 16\right)=8$ but $8\nmid 5$ so the equation $8x+16y=5$ has *no* integer solutions.

(b) The given statement:

If $d\mid a$, $d\mid b$ and $d\mid c$ then the Diophantine equation $ax+by=c$ has solutions.

This is also false.

For example, consider the Diophantine equation $8x+16y=14$. In this case $2\mid 8$, $2\mid 16$ and $2\mid 14$. However $\gcd\left(8,\quad 16\right)=8$ but $8\nmid 14$ so the equation $8x+16y=14$ has *no* integer solutions.

(c) This result is true. Here is the proof.

Proof.

The $\gcd\left(a,\quad a+1\right)=1$. *Why?*

Suppose $\gcd\left(a,\quad a+1\right)=g>1$ then

$$g\mid a \text{ and } g\mid\left(a+1\right) \text{ implies } g\mid\left(a+1-a\right) \text{ implies } g\mid 1$$

But our supposition is $g>1$. Contradiction, so $\gcd\left(a,\quad a+1\right)=1$. By the result of question 11:

> The $ax+by=1$ has a integer solution $\Leftrightarrow$ $\gcd\left(a,\quad b\right)=1$.

We have $ax+\left(a+1\right)y=1$ has integer solutions.

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