Complete Solutions to Supplementary Problems 1

1. We need to find $\gcd(69,\ 161)$. We can use the Euclidean algorithm to find this:

$$161 = (2\times 69) + \boxed{23}$$
$$69 = (3\times 23) + 0$$

Hence $\gcd(69,\ 161) = 23$. Factorizing each of these integers 69 and 161 into multiples of 23 we have

$$69 = 3\times 23 \text{ and } 161 = 7\times 23$$

Simplifying the given fraction $\dfrac{161}{69} = \dfrac{7\times \cancel{23}}{3\times \cancel{23}} = \dfrac{7}{3}$.

2. (i) First we need to find the $\gcd(57,\ 76)$. Applying the Euclidean algorithm we have

$$76 = (1\times 57) + \boxed{19} \qquad (*)$$
$$57 = (3\times 19) + 0$$

Therefore $\gcd(57,\ 76) = 19$. Now we need to solve the Diophantine equation

$$57x + 76y = \gcd(57,\ 76) = 19 \qquad (\dagger)$$

Rearranging (*) to make 19 the subject gives

$$19 = 76 - (57\times 1) = (76\times 1) + (57\times(-1))$$

From the last line we have $x = -1,\ \ y = 1$ as a solution to $57x + 76y = 19$.

(ii) We need to solve the given equation $57x + 76y = 95$. Note that

$$95 = 5\times 19$$

By part (i) we already have the solution for 19. Multiplying (†) by 5 gives

$$5\left[(76\times 1) + (57\times(-1))\right] = 5\times 19$$
$$(76\times 5) + (57\times(-5)) = 95$$

Hence $x = -5,\ \ y = 5$ is a solution to $57x + 76y = 95$.

3. (a) We need to solve $63x + 99y = \gcd(63,\ 99)$. Using the Euclidean algorithm to find the gcd we have

$$\begin{aligned} 99 &= \left(1\times 63\right)+36 \\ 63 &= \left(1\times 36\right)+27 \\ 36 &= \left(1\times 27\right)+\boxed{9} \\ 27 &= \left(3\times 9\right)+0 \end{aligned}$$

Therefore $\gcd\left(63,\quad 99\right)=9$ and we need to find x and y which satisfy

$$63x+99y=9$$

Using the first calculation backwards:

$$\begin{aligned} 9 &= 36-27 \\ &= 36-\left(63-36\right) \\ &= \left(2\times 36\right)-63 \\ &= \left(2\times\left[99-63\right]\right)-63 \\ &= \left(2\times 99\right)-\left(3\times 63\right)=99\left(2\right)+63\left(-3\right) \end{aligned}$$

Since $99\left(2\right)+63\left(-3\right)=9=\gcd\left(99,\quad 63\right)$ so an integer solution is $x=-3,\ \ y=2$.

(b) This time we need to solve $2014x+2015y=\gcd\left(2014,\quad 2015\right)$. Similarly, we have

$$2015=\left(1\times 2014\right)+1$$

Therefore $\gcd\left(2014,\quad 2015\right)=1$. Hence

$$2015-2014=2015\left(1\right)+2014\left(-1\right)=1$$

Our solution is $x=-1,\ \ y=1$.

(c) (i) We are asked to solve $2015x+39y=\gcd\left(2015,\quad 39\right)$. Determining the gcd by applying the Euclidean algorithm:

$$\begin{aligned} 2015 &= \left(51\times 39\right)+26 \\ 39 &= \left(1\times 26\right)+\boxed{13} \\ 26 &= \left(2\times 13\right)+0 \end{aligned}$$

We have $\gcd\left(2015,\quad 39\right)=13$. We need to solve the equation $2015x+39y=13$.

Retracing our footsteps in the above calculation yields:

$$\begin{aligned} 13=39-26 &= 39-\left[2015-\left(51\times 39\right)\right] \\ &= 52\left(39\right)-2015 \end{aligned}$$

Rewriting the last line as $39\left(52\right)+2015\left(-1\right)=13$. Hence $x=-1,\ \ y=52$.

(ii) Now we need to find solutions to

$$2015x+39y=-\gcd\left(2015,\quad 39\right)\underbrace{=}_{\text{by part (i)}}-13$$

Multiplying the calculation in part (i), $39(52)+2015(-1)=13$, by -1 gives

$$-\left[39(52)+2015(-1)\right]=-13$$
$$39(-52)+2015(1)=-13$$

Therefore $x=1,\ \ y=-52$.

4. (a) Applying the division algorithm to $a=37,\ \ b=4$ gives

$$37=(9\times 4)+1$$

The quotient is 9 and remainder is 1.

(b) This time let $a=-1007$ and $b=20$. Remember the remainder $r\geq 0$. Using a calculator we have $-51\times 20=-1020$. Since we want $a=-1007$ so

$$a=-1007=(-51\times 20)+13$$

The quotient is -51 and remainder is 13.

(c) We need to find the quotient and remainder of $-1\ 000\ 001$ divided by 999.

Again using a calculator we have

$$-1002\times 999=-1\ 000\ 998$$

Adding 997 to this gives

$$(-1002\times 999)+997=-1\ 000\ 998+997=-1\ 000\ 001$$

Therefore, the quotient is -1002 and remainder 997.

5. (a) The divisors of 100 are $\{\pm 1,\ \pm 2,\ \pm 4,\ \pm 5,\ \pm 10, \pm 20, \pm 25,\ \pm 50,\ \pm 100\}$.

(b) Same answer as part (a).

(c) The divisors of 200 are the divisors of 100 because 100 goes into 200 and ± 200 :

$$\{\pm 1,\ \pm 2,\ \pm 4,\ \pm 5,\ \pm 10, \pm 20, \pm 25,\ \pm 50,\ \pm 100,\ \pm 200\}$$

6. (a) Since $6\times 11=66$ so $6 \mid 66$. The given statement is true.

(b) As $6\times 0=0$ therefore $6 \mid 0$ is true.

(c) Clearly $7 \nmid 17$ so $7 \mid 17$ is false.

(d) Since $7\times 14=98$ so $7 \mid 98$ which implies $7 \nmid 98$ is false.

(e) The statement $0 \mid 7$ is false because every number multiplied by zero gives zero so cannot get 7. False.

7. *Can we find integers such that* $a \mid bc$ *but* $a \nmid b$ *and* $a \nmid c$ *?*

Yes. An example is $54 = 9\times 6$ and $27 \mid (9\times 6)$ but $27 \nmid 9$ and $27 \nmid 6$.

8. Two integers a and b are relatively prime if $\gcd(a, \ b) = 1$.

9. We are asked to show that $5 \mid (n^5 - n)$. *How?*

Use mathematical induction.

Step 1:

Clearly the result is true for $n = 1$ because

$$5 \mid (1^5 - 1) \quad \Rightarrow \quad 5 \mid 0$$

Step 2:

Assume the result is true for $n = k$:

$$5 \mid (k^5 - k) \qquad\qquad (*)$$

Step 3:

Required to prove the result is true for $n = k+1$. We need to show that

$$5 \mid \left((k+1)^5 - (k+1)\right) \qquad\qquad (**)$$

Expanding $(k+1)^5 - (k+1)$ by using the binomial expansion (I.37) on $(k+1)^5$ and then subtracting $k+1$ gives

$$\begin{aligned}(k+1)^5 - (k+1) &= (k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) - (k+1)\\ &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1\\ &= \underbrace{(k^5 - k)}_{5m \text{ by } (*)} + 5k^4 + 10k^3 + 10k^2 + 5k\\ &= 5m + 5k^4 + 10k^3 + 10k^2 + 5k = 5\left[m + k^4 + 2k^3 + 2k^2 + k\right]\end{aligned}$$

Therefore $5 \mid \left((k+1)^5 - (k+1)\right)$. By mathematical induction we have our result.

(You can also show that n and n^5 give the same remainder when dividing them by 5, that is why 5 divides their difference.)

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10. *Proof.*

Let n be any integer. By the Division Algorithm we have

$$n = 7q + r \qquad 0 \le r < 7$$

We examine the cube of this number n^3 by using the binomial formula:

(I.37) $$\left(a+b\right)^n = C_n a^n + C_{n-1} a^{n-1} b + C_{n-2} a^{n-2} b^2 + C_{n-3} a^{n-3} b^3 + \dots + C_0 b^n$$

We have

$$n^3 = \left(7q+r\right)^3 = \underbrace{\left(7q\right)^3 + 3\left(7q\right)^2 + 3\left(7q\right)}_{7m} + r^3 = 7m + r^3 \qquad \text{where } 0 \le r < 7$$

Remember the remainder r can only take values 0, 1, 2, 3, 4, 5 and 6. Finding the cube of each of these numbers and letting r' represent the remainder gives

$$0^3 = 0, \qquad r' = 0$$
$$1^3 = 1, \qquad r' = 1$$
$$2^3 = 8 = \left(1\times 7\right) - 1, \qquad r' = -1$$
$$3^3 = 27 = \left(4\times 7\right) - 1, \qquad r' = -1$$
$$4^3 = 64 = \left(9\times 7\right) + 1, \qquad r' = 1$$
$$5^3 = 125 = \left(18\times 7\right) - 1, \qquad r' = -1$$
$$6^3 = 216 = \left(31\times 7\right) - 1, \qquad r' = -1$$

r^3 can only have remainder values 0, 1 and -1 after dividing by 7. Hence the cube of any integer has the form $7k$ or $7k \pm 1$.

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11. The given statement $a \mid b$ and $b \mid a$ implies $a = b$ is false because

$$5 \mid -5 \quad \text{and} \quad -5 \mid 5 \text{ but } 5 \neq -5.$$

12. The given statement $a \nmid b$ and $a \mid bc \;\Rightarrow\; a \mid c$ is false because

$$8 \nmid 12 \quad \text{and} \quad 8 \mid \left(12\times 4\right) \quad \text{but} \quad 8 \nmid 4$$

It is true if a and b are relatively prime, that is

$$\gcd\left(a,\ b\right) = 1$$

This is Euclid's Lemma (1.13).

13. We are asked to prove $d \mid a$ and $d \mid b$ then $d^2 \mid ab$.

Proof.

We are given $d \mid a$ and $d \mid b$ so there are integers x and y such that

$$dx = a \quad \text{and} \quad dy = b$$

Multiplying these together gives

$$dx \times dy = d^2 xy = ab\,.$$

Since $d^2 xy = ab$ so by the definition of divisibility we have

$$d^2 \mid ab$$

This completes our proof.

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14. We need to prove that fourth power of an odd integer is of the form $16k+1$.

Proof.

Let n be an odd integer. By the division algorithm we can write

$$n = 4q + r \quad \text{where} \quad 0 \le r < 4 \qquad (*)$$

Taking the fourth power of this number (using binomial) yields

$$\begin{aligned} n^4 = (4q+r)^4 &= \underbrace{(4q)^4 + 4(4q)^3 r + 6(4q)^2 r^2 + 4(4q) r^3}_{=16m} + r^4 \\ &= 16m + r^4 \end{aligned}$$

Since we are considering an odd integer so the remainder r in (*) can only be $r=1$ or $r=3$. If $r=1$ then clearly we have our result. Putting $r=3$ into the above calculation gives

$$\begin{aligned} n^4 = (4q+r)^4 = 16m + 3^4 &= 16m + 81 \\ &= 16m + (5 \times 16) + 1 \\ &= 16(m+5) + 1 = 16k + 1 \quad \text{where } k = m+5 \end{aligned}$$

Hence the fourth power of an odd integer is of the form $16k+1$.

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15. We need to show that there are no integer solutions to $6x + 30y = 4$. First, we find the gcd of 6 and 30:

$$\gcd(6,\ 30) = 6$$

Since the right-hand side of the given equation is *not* a multiple of 6 so there are no integer solutions to the equation.

16. Clearly $-2 \mid 4$ but gcd must be positive (by definition). Therefore $\gcd(-2,\ 4) = 2$ not -2.

17. The error is in step B because we have omitted the binomial coefficients in the expansion of $4q+r$. It should be

$$\left(4q+r\right)^4=\left(4q\right)^4+4\left(4q\right)^3r+6\left(4q\right)^2r^2+4\left(4qr^3\right)+r^4$$

18. We are asked to find $\gcd\left(1\ 000\ 001,\quad 1\ 122\ 211\right)$. We apply the Euclidean algorithm:

$$\begin{aligned}1\ 122\ 211&=\left(1\times 1\ 000\ 001\right)+122\ 210\\1\ 000\ 001&=\left(8\times 122\ 210\right)+22\ 321\\122\ 210&=\left(5\times 22\ 321\right)+10\ 605\\22\ 321&=\left(2\times 10\ 605\right)+1111\\10\ 605&=\left(9\times 1111\right)+606\\1111&=\left(1\times 606\right)+505\\606&=\left(1\times 505\right)+\boxed{101}\\505&=\left(5\times 101\right)+0\end{aligned}$$

Hence $\gcd\left(1\ 000\ 001,\quad 1\ 122\ 211\right)=101$.

(ii) By Proposition (1.10) part (ii):

The $\gcd\left(a,\quad b\right)=g$ is the least positive integer value of $ma+nb$ where m and n range over all the integers.

We have the smallest positive integer of $1\ 000\ 001x+1\ 122\ 211y$ is 101 because by part (i) we have $\gcd\left(1\ 000\ 001,\quad 1\ 122\ 211\right)=101$.

(iii) We are asked to find to solve $1\ 000\ 001x+1\ 122\ 211y=202$. Note that $202=2\times 101$. Using the Euclidean algorithm derivation in part (i) backwards we have

$$\begin{aligned}101 &= 606 - 505\\
&= 606 - \left[1111 - \left(1\times 606\right)\right]\\
&= 2\left(606\right) - 1111\\
&= 2\left(10\,605 - \left(9\times 1111\right)\right) - 1111\\
&= 2\left(10605\right) - 19\left(1111\right)\\
&= 2\left(10605\right) - 19\left[22\,321 - \left(2\times 10\,605\right)\right]\\
&= 40\left(10605\right) - 19\left[22\,321\right]\\
&= 40\left[122\,210 - \left(5\times 22\,321\right)\right] - 19\left[22\,321\right]\\
&= 40\left[122\,210\right] - 219\left[22\,321\right]\\
&= 40\left[122\,210\right] - 219\left[1\,000\,001 - \left(8\times 122\,210\right)\right]\\
&= 1792\left[122\,210\right] - 219\left[1\,000\,001\right]\\
&= 1792\left[1\,122\,211 - \left(1\times 1\,000\,001\right)\right] - 219\left[1\,000\,001\right]\\
&= 1792\left[1\,122\,211\right] - 2011\left[1\,000\,001\right]\end{aligned}$$

We have $1792\left[1\,122\,211\right] - 2011\left[1\,000\,001\right] = 101$ which we can rewrite as

$$1\,122\,211\left[1792\right] + 1\,000\,001\left[-2011\right] = 101$$

For the equation $1\,000\,001x + 1\,122\,211y = 101$ we have the solution

$$x_0 = -2011 \text{ and } y_0 = 1792$$

Since we are given $1\,000\,001x + 1\,122\,211y = 202 = 2\times 101$ so our solution is

$$x = -2011\times 2 = -4022 \text{ and } y = 1792\times 2 = 3584$$

19. *Proof.*

By Proposition (1.17):

The Diophantine equation $ax + by = c$ has integer solutions $\Leftrightarrow g \mid c$.

We are given $\gcd\left(a, \quad b\right) = 1$ and $1 \mid c$ so $ax + by = c$ has integer solutions.

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20. We are asked to find the general solution of $2014x + 2015y = 2016$. Since

$$\gcd\left(2014, \quad 2015\right) = 1$$

By the result of the previous question we have integer solutions. We can first solve

$$2014x + 2015y = 1$$

An integer solution to this is $x_0 = -1,\ y = 1$. Now using Corollary (1.19):

All solutions of $ax + by = c$ are given by

$$x = x_0 + bt \text{ and } y = y_0 - at$$

In our case $a = 2014$ and $b = 2015$. Substituting $x_0 = -1,\ y_0 = 1,\ a = 2014,$ $b = 2015$ into $x = x_0 + bt$ and $y = y_0 - at$ gives

$$x = -1 + 2015t = 2015t - 1 \text{ and } y = 1 - 2014t$$

where t is any integer.

A particular solution for $t = 1$ is

$$x = (2015 \times 1) - 1 = 2014 \text{ and } y = 1 - (2014 \times 1) = -2013.$$

We have $x = 2014$ and $y = -2013$.

21. Let f be the number of fish sold and c be the number of portions of chips sold. We have the Diophantine equation

$$2.80f + 0.9c = 200$$

Multiplying this equation by 10 yields

$$28f + 9c = 2000 \qquad (\ddagger)$$

We need to find the $\gcd(28,\ 9)$ by using the Euclidean algorithm:

$$28 = (3 \times 9) + 1$$

Hence $\gcd(28,\ 9) = 1$. By using the above calculation we can find the solution to

$$28f + 9c = 1 \qquad (\dagger)$$

Rearranging $28 = (3 \times 9) + 1$ gives $28 - (3 \times 9) = 28(1) + 9(-3) = 1$. Equation $(\dagger)$ has the solution $f' = 1,\ c' = -3$. Multiplying this by 2000 gives the solutions

$$f_0 = 1 \times 2000 = 2000,\ c_0 = -3 \times 2000 = -6000 \text{ to equation } (\ddagger).$$

Applying Corollary (1.19):

> *All* solutions of $af + bc = d$ are given by
> $$f = f_0 + bt \text{ and } c = c_0 - at.$$

With $f_0 = 2000,\ c_0 = -6000,\ a = 28,\ b = 9$:

$$f = 2000 + 9t \text{ and } c = -6000 - 28t$$

The number of portion of chips sold cannot be negative so we must have

$$c = -6000 - 28t \geq 0 \quad \Leftrightarrow \quad -28t \geq 6000 \quad \Leftrightarrow \quad t \leq -\frac{6000}{28} = -214.28$$

We also cannot have negative number of fish sold. Therefore

$$f = 2000 + 9t \geq 0 \quad \Leftrightarrow \ 9t \geq -2000 \quad \Leftrightarrow \quad t \geq -\frac{2000}{9} = -222.22$$

Combining these two inequalities, $t \leq -214.28$ and $t \geq -222.22$, gives

$$-222.22 \le t \le -214.28$$

We are dealing with integers so the only values t can take are

$$t = -215,\ -216,\ -217,\ -218,\ -219,\ -220,\ -221 \text{ and } -222$$

(i) The largest feasible value for t is -215. Substituting $t = -215$ into

$$f = 2000 + 9t \text{ and } c = -6000 - 28t$$

gives $f = 2000 + 9t = 2000 + \left[9 \times \left(-215\right)\right] = 65$ and

$$c = -6000 - 28t = -6000 - \left[28 \times \left(-215\right)\right] = 20$$

The least portion of chips sold is 20.

(ii) The smallest feasible value for $t = -222$. Substituting this into the above gives

$$f = 2000 + 9t = 2000 + \left[9 \times \left(-222\right)\right] = 2$$

$$c = -6000 - 28t = -6000 - \left[28 \times \left(-222\right)\right] = 216$$

The largest portion of chips sold are 216.

22. We are asked to prove that if $a \nmid c$ but $a \mid (bc)$ then $\gcd(a,\ b) > 1$.

Proof.

This is straightforward because it is contrapositive statement of Euclid's Lemma (1.13):

$$\boxed{\text{If } a \mid (bc) \text{ with } \gcd(a,\ b) = 1 \text{ then } a \mid c.}$$

Recall the contrapositive of this says if $a \nmid c$ and $a \mid (bc)$ then $\gcd(a,\ b) \ne 1$ which implies that $\gcd(a,\ b) > 1$ because by the definition of $\gcd(a,\ b) \ge 1$.

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23. We are asked to prove $a^n \mid b^n$ implies $a \mid b$.

Proof.

Suppose $a \nmid b$. Therefore, for every integer x we have $ax \ne b$. This implies that for every integer y we have $a^n y \ne b^n$ which in turn implies $a^n \nmid b^n$. This is a contradiction because we are given $a^n \mid b^n$. Hence

$$a^n \mid b^n \text{ implies } a \mid b.$$

This completes our proof.

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24. We are asked to prove the following:

If $a \mid (b_1 \times b_2 \times \cdots \times b_n)$ and

$$\gcd(a,\ b_1) = \gcd(a,\ b_2) = \cdots = \gcd(a,\ b_{n-1}) = 1 \text{ (pairwise prime)}$$

Then $a \mid b_n$.

How do we prove this?

By applying mathematical induction.

Proof.

For the base case $n = 2$ we have our result by Euclid's Lemma (1.13):

If $x \mid (y \times z)$ with $\gcd(x,\ y) = 1$ then $x \mid z$.

That is $a \mid (b_1 \times b_2)$ and $\gcd(a,\ b_1) = 1$ implies that $a \mid b_2$.

Assume the result is true for $n = k$;

If $a \mid (b_1 \times b_2 \times \cdots \times b_k)$ and

$$\gcd(a,\ b_1) = \gcd(a,\ b_2) = \cdots = \gcd(a,\ b_{k-1}) = 1 \text{ (pairwise prime)}$$

Then $a \mid b_k$.

Consider the case $n = k + 1$. We need to show that:

If $a \mid (b_1 \times b_2 \times \cdots \times b_k \times b_{k+1})$ and

$$\gcd(a,\ b_1) = \gcd(a,\ b_2) = \cdots = \gcd(a,\ b_k) = 1 \text{ (pairwise prime).}$$

Then $a \mid b_{k+1}$.

From the result of question 15(ii) Exercises 1.3 we have

$$\gcd(a,\ b_1 \times b_2 \times \cdots \times b_k) = 1.$$

From $a \mid (b_1 \times b_2 \times \cdots \times b_k \times b_{k+1})$ we have

$$a \mid ([b_1 \times b_2 \times \cdots \times b_k] \times b_{k+1}).$$

Applying Euclid's Lemma on this with $\gcd(a,\ b_1 \times b_2 \times \cdots \times b_k) = 1$ gives

$$a \mid b_{k+1}.$$

Hence by mathematical induction we have our result;

If $a \mid (b_1 \times b_2 \times \cdots \times b_n)$ and

$$\gcd(a,\ b_1) = \gcd(a,\ b_2) = \cdots = \gcd(a,\ b_{n-1}) = 1 \text{ (pairwise prime).}$$

Then $a \mid b_n$. This completes our proof.

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