11장 연습문제 정답

[Section 11.1]

1.

(a) 참

(b) 거짓 / 특잇값 분해는 모든 행렬에 적용 가능하다.

(c) 거짓 / 특잇값 분해는 행렬의 크기에 무관하게 적용 가능하다.

(d) 참

(e) 참

(f) 거짓 / 복수 개의 특잇값 분해가 가능한 경우도 있다.

(g) 참

2.

(a) 3, 1

(b) $\sqrt{3}$, 1, 0

(c) 5, 0

3.

(a) $A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}$

(b) $A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

(c) $A = U\Sigma V^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{-1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix}$

(d) $A = U\Sigma V^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

(e) $A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{11} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{22}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{11}} \\ \frac{2}{\sqrt{22}} & 0 & -\frac{3}{\sqrt{11}} \\ \frac{3}{\sqrt{22}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{11}} \end{bmatrix}$

(f) $A = U\Sigma V^T = \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

(g) $A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

4.

(a) 3

(b) 2

5.

$A = 7\begin{bmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix} + 6\begin{bmatrix} -\frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{bmatrix}$

6.

$A = 2\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} + \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

7.

A를 $A = U\Sigma V^{\top}$으로 특잇값 분해할 때, V의 열벡터 v_1과 v_2를 사용하여 단위 벡터 u를 $u = c_1 v_1 + c_2 v_2$로 표현해보자. 이때 $\|u\|^2 = c_1^2 + c_2^2 = 1$이다.

이제 $Au = c_1 Av_1 + c_2 Av_2$에 대해 노름을 구하면 다음과 같다.

$$\begin{aligned} \|Au\|^2 &= c_1^2\|Av_1\|^2 + c_2^2\|Av_2\|^2 \\ &= c_1^2\sigma_1^2 + c_2^2\sigma_2^2 \\ &\le (c_1^2 + c_2^2)\sigma_1^2 \quad (\because \sigma_1 \ge \sigma_2) \\ &= \sigma_1^2 \quad (\because c_1^2 + c_2^2 = 1) \end{aligned}$$

마찬가지로 다음의 관계가 성립한다.

$$
\begin{aligned}
\|A\boldsymbol{u}\|^2 &= c_1^2\|A\boldsymbol{v}_1\|^2 + c_2^2\|A\boldsymbol{v}_2\|^2 \\
&= c_1^2\sigma_1^2 + c_2^2\sigma_2^2 \\
&\geq (c_1^2+c_2^2)\sigma_2^2 \quad (\because \sigma_1 \geq \sigma_2) \\
&= \sigma_2^2 \quad (\because c_1^2+c_2^2=1)
\end{aligned}
$$

따라서 $\sigma_2^2 \leq \|A\boldsymbol{u}\|^2 \leq \sigma_1^2$이다.

8.

(a) $\sqrt{10}$, 0

(b) $2\sqrt{5}$, 0, 0

9.

(a) $A=\begin{bmatrix}-1 & -i \\ -i & -1\end{bmatrix}=\begin{bmatrix}-\dfrac{i}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \\ -\dfrac{1}{\sqrt{2}} & -\dfrac{i}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0 \\ 0 & \sqrt{2}\end{bmatrix}\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}$

(b) $A=\begin{bmatrix}-1 & -i & 1 \\ -i & -1 & i \\ 1 & -i & 1\end{bmatrix}=\begin{bmatrix}\dfrac{1}{\sqrt{2}} & -\dfrac{i}{\sqrt{6}} & -\dfrac{1}{\sqrt{3}} \\ \dfrac{i}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} & \dfrac{i}{\sqrt{3}} \\ 0 & -i\sqrt{\dfrac{2}{3}} & \dfrac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-\dfrac{1}{\sqrt{2}} & 0 & \dfrac{1}{\sqrt{2}} \\ -\dfrac{i}{\sqrt{6}} & \sqrt{\dfrac{2}{3}} & -\dfrac{i}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}} & -\dfrac{i}{\sqrt{3}} & \dfrac{1}{\sqrt{3}}\end{bmatrix}$

[Section 11.2]

10.

(a) 참

(b) 참

(c) 거짓 / 행렬의 의사 역행렬은 유일하다.

(d) 참

(e) 참

11.

(a) $A^+=\dfrac{1}{6}\begin{bmatrix}-5 & 2 \\ 1 & 2 \\ -2 & 2\end{bmatrix}$

(b) $A^+=\begin{bmatrix}\dfrac{-1}{2\sqrt{2}} & \dfrac{1}{4} \\ \dfrac{1}{2\sqrt{2}} & \dfrac{1}{4}\end{bmatrix}$

12.

(a) $\begin{bmatrix} 0 \\ 3/2 \end{bmatrix}$

(b) $\begin{bmatrix} 6/5 \\ -8/5 \end{bmatrix}$

13.

(a) $\widehat{A_2} \approx \begin{bmatrix} -0.14 & 0.52 & 1.82 & 1.02 & 4.30 \\ 2.78 & 2.25 & 2.11 & 3.24 & 0.31 \\ 0.69 & 1.05 & 2.01 & 1.72 & 3.39 \\ 2.29 & 1.71 & 1.32 & 2.41 & -0.68 \\ 1.29 & 1.80 & 3.27 & 2.90 & 5.28 \end{bmatrix}$

(b) $\widehat{A_2} \approx \begin{bmatrix} 0.91 & 1.13 & 0.89 & 0.07 & -0.07 \\ 2.74 & 3.40 & 2.68 & 0.22 & -0.20 \\ 1.65 & 2.29 & 2.03 & 1.23 & 1.09 \\ 2.77 & 3.52 & 2.84 & 0.57 & 0.19 \\ 0.17 & 1.16 & 1.76 & 4.26 & 4.70 \\ -0.32 & 0.66 & 1.47 & 4.72 & 5.30 \end{bmatrix}$

14.

$x_1 = 0.18, x_2 = -0.66, x_3 = 0.06, x_4 = 6.18$

15.

$x_1 = -2.70, x_2 = 0.22, x_3 = -2.03, x_4 = 2.05, x_5 = 2.37$