Hints and Answers

SECTION 1-3

2. a. $\alpha(t) = (t - \sin t, 1 - \cos t)$; see Fig. 1-7. Singular points: $t = 2\pi n$, where n is any integer.

7. b. Apply the mean value theorem to each of the functions x, y, z to prove that the vector $(\alpha(t+h) - \alpha(t+k))/(h-k)$ converges to the vector $\alpha'(t)$ as $h, k \to 0$. Since $\alpha'(t) \neq 0$, the line determined by $\alpha(t+h)$, $\alpha(t+k)$ converges to the line determined by $\alpha'(t)$.

8. By the definition of integral, given $\epsilon > 0$, there exists a $\delta' > 0$ such that if $|P| < \delta'$, then

$$\left| \left(\int_a^b |\alpha'(t)| dt \right) - \sum (t_i - t_{i-1}) |\alpha'(t_i)| \right| < \frac{\epsilon}{2}.$$

On the other hand, since α' is uniformly continuous in $[a, b]$, given $\epsilon > 0$, there exists $\delta'' > 0$ such that if $t, s \in [a, b]$ with $|t - s| < \delta''$, then

$$|\alpha'(t) - \alpha'(s)| < \epsilon/2(b-a).$$

Set $\delta = \min(\delta', \delta'')$. Then if $|P| < \delta$, we obtain, by using the mean value theorem for vector functions,

$$\begin{aligned}
&\left| \sum |\alpha(t_{i-1}) - \alpha(t_i)| - \sum (t_{i-1} - t_i) |\alpha'(t_i)| \right| \\
&\leq \left| \sum (t_{i-1} - t_i) \sup_{s_i} |\alpha'(s_i)| - \sum (t_{i-1} - t_i) |\alpha'(t_i)| \right| \\
&\leq \left| \sum (t_{i-1} - t_i) \sup_{s_i} |\alpha'(s_i) - \alpha'(t_i)| \right| \leq \frac{\epsilon}{2},
\end{aligned}$$

where $t_{i-1} \le s_i \le t_i$. Together with the above, this gives the required inequality.

SECTION 1-4

2. Let the points $p_0 = (x_0, y_0, z_0)$ and $p = (x, y, z)$ belong to the plane P. Then $ax_0 + by_0 + cz_0 + d = 0 = ax + by + cz + d$. Thus, $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. Since the vector $(x - x_0, y - y_0, z - z_0)$ is parallel to P, the vector (a, b, c) is normal to P. Given a point $p = (x, y, z) \in P$, the distance ρ from the plane P to the origin O is given by $\rho = |p| \cos\theta = (p \cdot v)/|v|$, where θ is the angle of Op with the normal vector v. Since $p \cdot v = -d$,

$$\rho = \frac{p \cdot v}{|v|} = -\frac{d}{|v|}.$$

3. This is the angle of their normal vectors.

4. Two planes are parallel if and only if their normal vectors are parallel.

6. v_1 and v_2 are both perpendicular to the line of intersection. Thus, $v_1 \wedge v_2$ is parallel to this line.

7. A plane and a line are parallel when a normal vector to the plane is perpendicular to the direction of the line.

8. The direction of the common perpendicular to the given lines is the direction of $u \wedge v$. The distance between these lines is obtained by projecting the vector $r = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$ onto the common perpendicular. Such a projection is clearly the inner product of r with the unit vector $(u \wedge v)/|u \wedge v|$.

SECTION 1-5

2. Use the fact that $\alpha' = t$, $\alpha'' = kn$, $\alpha''' = kn' + k'n = -k^2 t + k'n - k\tau b$.

4. Differentiate $\alpha(s) + \lambda(s)n(s) = \text{const.}$, obtaining

$$(1 - \lambda k)t + \lambda' n - \lambda \tau b = 0.$$

It follows that $\tau = 0$ (the curve is contained in a plane) and that $\lambda = \text{const.} = 1/k$.

7. **a.** Parametrize α by arc length.

b. Parametrize α by arc length s. The normal lines at s_1 and s_2 are

$$\beta_1(t) = \alpha(s_1) + tn(s_1), \quad \beta_2(\tau) = \alpha(s_2) + \tau n(s_2), \quad t \in R, \tau \in R,$$

respectively. Their point of intersection will be given by values of t and τ such that

$$\frac{\alpha(s_2)-\alpha(s_1)}{s_2-s_1}=\frac{tn(s_1)-\tau n(s_2)}{s_2-s_1}.$$

Take the inner product of the above with $\alpha'(s_1)$ to obtain $1 = (-\lim \tau)_{s_2\to s_1}\cdot\langle\alpha'(s_1), n'(s_1)\rangle$. It follows that τ converges to $1/k$ as $s_2 \to s_1$.

13. To prove that the condition is necessary, differentiate three times $|\alpha(s)|^2 = \text{const.}$, obtaining $\alpha(s) = -Rn + R'Tb$. For the sufficiency, differentiate $\beta(s) = \alpha(s) + Rn - R'Tb$, obtaining

$$\beta'(s) = t + R(-kt - \tau b) + R'n - (TR')'b - Rn = -(R'\tau + (TR')')b.$$

On the other hand, by differentiating $R^2 + (TR')^2 = \text{const.}$, one obtains

$$0 = 2RR' + 2(TR')(TR') = \frac{2R'}{\tau}(R\tau + (TR')'),$$

since $k' \neq 0$ and $\tau \neq 0$. Hence, $\beta(s)$ is a constant p_0, and

$$|\alpha(s) - p_0|^2 = R^2 + (TR')^2 = \text{const.}$$

15. Since $b' = \tau n$ is known, $|\tau| = |b'|$. Then, up to a sign, n is determined. Since $t = n \wedge b$ and the curvature is positive and given by $t' = kn$, the curvature can also be determined.

16. First show that

$$\frac{n\wedge n'\cdot n''}{|n'|^2}=\frac{\left(\dfrac{k}{\tau}\right)'}{\left(\dfrac{k}{\tau}\right)^2+1}=a(s).$$

Thus, $\int a(s)\,ds = \operatorname{arc\,tan}(k/\tau)$; hence, k/τ can be determined; since k is positive, this also gives the sign of τ. Furthermore, $|n'|^2 = |-kt - \tau b|^2 = k^2 + \tau^2$ is also known. Together with k/τ, this suffices to determine k^2 and τ^2.

17. a. Let a be the unit vector of the fixed direction and let θ be the constant angle. Then $t\cdot a = \cos\theta = \text{const.}$, which differentiated gives $n\cdot a = 0$. Thus, $a = t\cos\theta + b\sin\theta$, which differentiated gives $k\cos\theta + \tau\sin\theta = 0$, or $k/\tau = -\tan\theta = \text{const.}$ Conversely, if $k/\tau = \text{const.} = -\tan\theta = -(\sin\theta/\cos\theta)$, we can retrace our steps, obtaining that $t\cos\theta + b\sin\theta$ is a constant vector a. Thus, $t \cdot a = \cos\theta = \text{const.}$

b. From the argument of part a, it follows immediately that $t \cdot a = \text{const.}$ implies that $n \cdot a = 0$; the last condition means that n is parallel to a plane normal to a. Conversely, if $n \cdot a = 0$, then $(dt/ds) \cdot a = 0$; hence, $t \cdot a = \text{const.}$

c. From the argument of part a, it follows that $t \cdot a = \text{const.}$ implies that $b \cdot a = \text{const.}$ Conversely, if $b \cdot a = \text{const.}$, by differentiation we find that $n \cdot a = 0$.

18. a. Parametrize α by arc length s and differentiate $\bar{\alpha} = \alpha + rn$ with respect to s, obtaining

$$\frac{d\bar{\alpha}}{ds} = (1 - rk)t + r'n - r\tau b.$$

Since $d\bar{\alpha}/ds$ is tangent to $\bar{\alpha}$, $(d\bar{\alpha}/ds) \cdot n = 0$; hence, $r' = 0$.

b. Parametrize α by arc length s, and denote by $\bar{s}$ and $\bar{t}$ the arc length and the unit tangent vector of $\bar{\alpha}$. Since $d\bar{t}/ds = (d\bar{t}/d\bar{s})(d\bar{s}/ds)$, we obtain that

$$\frac{d}{ds}(t \cdot \bar{t}) = t \cdot \frac{d\bar{t}}{ds} + \frac{dt}{ds} \cdot \bar{t} = 0;$$

hence, $t \cdot \bar{t} = \text{const.} = \cos\theta$. Thus, by using that $\bar{\alpha} = \alpha + rn$, we have

$$\cos\theta = \bar{t} \cdot t = \frac{d\bar{\alpha}}{ds}\frac{ds}{d\bar{s}} \cdot t = \frac{ds}{d\bar{s}}(1 - rk),$$

$$|\sin\theta| = |\bar{t} \wedge t| = \left|\frac{ds}{d\bar{s}}((t + rn') \wedge t\right| = \left|\frac{ds}{d\bar{s}} r\tau\right|.$$

From these two relations, it follows that

$$\frac{1 - rk}{r\tau} = \text{const.} = \frac{B}{r}.$$

Thus, setting $r = A$, we finally obtain that $Ak + B\tau = 1$.

Conversely, let this last relation hold, set $A = r$, and define $\bar{\alpha} = \alpha + rn$. Then, by again using the relation, we obtain

$$\frac{d\bar{\alpha}}{ds} = (1 - rk)t - r\tau b = \tau(Bt - rb).$$

Thus, a unit vector $\bar{t}$ of $\bar{\alpha}$ is $(Bt - rb)/\sqrt{B^2 + r^2} = \bar{t}$. It follows that $d\bar{t}/ds = ((Bk - r\tau)/\sqrt{B^2 + r^2})n$. Therefore, $\bar{n}(s) = \pm n(s)$ and the normal lines of $\bar{\alpha}$ and α at s agree. Thus, α is a Bertrand curve.

c. Assume the existence of two distinct Bertrand mates $\bar{\alpha} = \alpha + \bar{r}n$, $\tilde{\alpha} = \alpha + \tilde{r}n$. By part b there exist constants c_1 and c_2 so that $1 - \bar{r}k = c_1(\bar{r}\tau)$, $1 - \bar{r}k = c_2(\bar{r}\tau)$. Clearly, $c_1 \neq c_2$. Differentiating these expressions, we obtain $k' = \tau' c_1$, $k' = \tau' c_2$, respectively. This implies that $k' = \tau' = 0$. Using the uniqueness part of the fundamental theorem of the local theory of curves, it is easy to see that the circular helix is the only such curve.

SECTION 1-6

1. Assume that $s = 0$, and consider the canonical form around $s = 0$. By condition 1, P must be of the form $z = cy$, or $y = 0$. The plane $y = 0$ is the rectifying plane, which does not satisfy condition 2. Observe now that if $|s|$ is sufficiently small, $y(s) > 0$, and $z(s)$ has the same sign as s. By condition 2, $c = z/y$ is simultaneously positive and negative. Thus, P is the plane $z = 0$.

2. a. Consider the canonical form of $\alpha(s) = (x(s), y(s), z(s))$ in a neighborhood of $s = 0$. Let $ax + by + cz = 0$ be the plane that passes through $\alpha(0)$, $\alpha(0 + h_1)$, $\alpha(0 + h_2)$. Define a function $F(s) = ax(s) + by(s) + cz(s)$ and notice that $F(0) = F(h_1) = F(h_2) = 0$. Use the canonical form to show that $F'(0) = a$, $F''(0) = bk$. Use the mean value theorem (twice) to show that as $h_1, h_2 \to 0$, then $a \to 0$ and $b \to 0$. Thus, as $h_1, h_2 \to 0$ the plane $ax + by + cz = 0$ approaches the plane $z = 0$, that is, the osculating plane.

SECTION 1-7

1. No. Use the isoperimetric inequality.

2. Let S^1 be a circle such that $\overline{AB}$ is a chord of S^1 and one of the two arcs α and β determined by A and B on S^1, say α, has length l. Consider the piecewise C^1 closed curve (see Remark 2 after Theorem 1) formed by β and C. Let β be fixed and C vary in the family of all curves joining A to B with length l. By the isoperimetric inequality for piecewise C^1 curves, the curve of the family that bounds the largest area is S^1. Since β is fixed, the arc of circle α is the solution to our problem.

4. Choose coordinates such that the center O is at p and the x and y axes are directed along the tangent and normal vectors at p, respectively. Parametrize C by arc length, $\alpha(s) = (x(s), y(s))$, and assume that $\alpha(0) = p$. Consider the (finite) Taylor's expansion

$$\alpha(s) = \alpha(0) + \alpha'(0)s + \alpha''(0)\frac{s^2}{2} + R,$$

where $\lim_{s\to 0} R/s^2 = 0$. Let k be the curvature of α at $s = 0$, and obtain

$$x(s) = s + R_x, \quad y(s) = \pm\frac{ks^2}{2} + R_y,$$

where $R = (R_x, R_y)$ and the sign depends on the orientation of α. Thus,

$$|k| = \lim_{s\to 0} \frac{2|y(s)|}{s^2} = \lim_{d\to 0} \frac{2h}{d^2}.$$

5. Let O be the center of the disk D. Shrink the boundary of D through a family of concentric circles until it meets the curve C at a point p. Use Exercise 4 to show that the curvature k of C at p satisfies $|k| \geq 1/r$.

8. Since α is simple, we have, by the theorem of turning tangents,

$$\int_0^t k(s)\, ds = \theta(l) - \theta(0) = 2\pi.$$

Since $k(s) \leq c$, we obtain

$$2\pi = \int_0^l k(s)\, ds \leq c\int_0^l ds = cl.$$

9. We first observe that the intersection of convex sets is a convex set. Since the curve is convex, each tangent line determines a half-plane that contains the curve. The intersection all such half-planes is a convex set K' which contains the set K bounded by the curve. Also $K' \subset K$, for if $q' \subset K'$, $q' \notin K$, the segment $q'p'$, $q' \in K'$, $p' \in K \subset K'$ is contained in K' by convexity, and meets the curve. This is easily seen to yield a contradiction.

11. Observe that the area bounded by H is greater than or equal to the area bounded by C and that the length of H is smaller than or equal to the length of C. Expand H through a family of curves parallel to H (Exercise 6) until its length reaches the length of C. Since the area either remains the same or has been further increased in this process, we obtain a convex curve H' with the same length as C but bounding an area greater than or equal to the area of C.

12.

$$M_1 = \int_0^{2\pi}\left(\int_0^{1/2} dp\right) d\theta = \pi,$$

$$M_2 = \int_0^{2\pi}\left(\int_0^{1} dp\right) d\theta = 2\pi.$$

(See Fig. 1-40.) Thus, $M_1/M_2 = \frac{1}{2}$.

SECTION 2-2

5. Yes.

11. b. To see that $\mathbf{x}$ is one-to-one, observe that from z one obtains $\pm u$. Since $\cosh v > 0$, the sign of u is the same as the sign of x. Thus, $\sinh v$ (and hence v) is determined.

13. $\mathbf{x}(u, v) = (\sinh u \cos v, \sinh u \sin v, \cosh v)$.

15. Eliminate t in the equations $x/a = y/t = -(z-t)/t$ of the line joining $p(t) = (0, 0, t)$ to $q(t) = (a, t, 0)$.

17. c. Extend Prop. 3 for plane curves and apply the argument of Example 5.

18. For the first part, use the inverse function theorem. To determine F, set $u = \rho^2$, $v = \tan\varphi$, $w = \tan^2\theta$. Write $x = f(\rho, \theta)\cos\varphi$, $y = f(\rho, \theta)\sin\varphi$, where f is to be determined. Then

$$x^2 + y^2 + z^2 = f^2 + z^2 = \rho^2, \qquad \frac{f^2}{z^2} = \tan^2\theta.$$

It follows that $f = \rho\sin\theta$, $z = \rho\cos\theta$. Therefore,

$$F(u, v, w) = \left(\frac{\sqrt{uw}}{\sqrt{(1+w)(1+v^2)}}, \frac{v\sqrt{uw}}{\sqrt{(1+w)(1+v^2)}}, \frac{\sqrt{u}}{\sqrt{1+w}}\right).$$

19. No. For C, observe that no neighborhood in R^2 of a point in the vertical arc can be written as the graph of a differentiable function. The same argument applies to S.

SECTION 2-3

1. Since $A^2 =$ identity, $A = A^{-1}$.

5. d is the restriction to S of a function $d\colon R^3 \to R$:

$$d(x, y, z) = \{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\}^{1/2},$$
$$(x, y, z) \neq (x_0, y_0, z_0).$$

8. If $p = (x, y, z)$, $F(p)$ lies in the intersection with H of the line $t \to (tx, ty, z)$, $t > 0$. Thus,

$$F(p) = \left(\frac{\sqrt{1+z^2}}{\sqrt{x^2+y^2}}x, \frac{\sqrt{1+z^2}}{\sqrt{x^2+y^2}}y, z\right).$$

Let U be R^3 minus the z axis. Then $F\colon U \subset R^3 \to R^3$ as defined above is differentiable.

13. If f is such a restriction, f is differentiable (Example 1). To prove the converse, let $\mathbf{x}\colon U \to R^3$ be a parametrization of S in p. As in Prop. 1, extend $\mathbf{x}$ to $F\colon U \times R \to R^3$. Let W be a neighborhood of p in R^3 on which F^{-1} is a diffeomorphism. Define $g\colon W \to R$ by $g(q) = f \circ \mathbf{x} \circ \pi \circ F^{-1}(q)$, $q \in W$, where $\pi\colon U \times R \to U$ is the natural projection. Then g is differentiable, and the restriction $g|W \cap S = f$.

16. F is differentiable in $S^2 - \{N\}$ as a composition of differentiable maps. To prove that F is differentiable at N, consider the stereographic projection π_S from the south pole $S = (0, 0, -1)$ and set $Q = \pi_S \circ F \circ \pi_S^{-1}\colon U \subset \mathbb{C} \to \mathbb{C}$ (of course, we are identifying the plane $z = 1$ with $\mathbb{C}$). Show that $\pi_N \circ \pi_S^{-1}\colon \mathbb{C} - \{0\} \to \mathbb{C}$ is given by $\pi_N \circ \pi_S^{-1}(\zeta) = 1/\bar{\zeta}$. Conclude that

$$Q(\zeta) = \frac{\zeta^n}{\bar{a}_0 + \bar{a}_1\zeta + \cdots + \bar{a}_n\zeta^n};$$

hence, Q is differentiable at $\zeta = 0$. Thus, $F = \pi_S^{-1} \circ Q \circ \pi_S$ is differentiable at N.

SECTION 2-4

1. Let $\alpha(t) = (x(t), y(t), z(t))$ be a curve on the surface passing through $p_0 = (x_0, y_0, z_0)$ for $t = 0$. Thus, $f(x(t), y(t), z(t)) = 0$; hence, $f_x x'(0) + f_y y'(0) + f_z z'(0) = 0$, where all derivatives are computed at p_0. This means that all tangent vectors at p_0 are perpendicular to the vector (f_x, f_y, f_z), and hence the desired equation.

4. Denote by f' the derivative of $f(y/x)$ with respect to $t = y/x$. Then $z_x = f - (y/x)f'$, $z_y = f'$. Thus, the equation of the tangent plane at (x_0, y_0) is $z = x_0 f + (f - (y_0/x_0)f')(x - x_0) + f'(y - y_0)$, where the functions are computed at (x_0, y_0). It follows that if $x = 0$, $y = 0$, then $z = 0$.

12. For the orthogonality, consider, for instance, the first two surfaces. Their normals are parallel to the vectors $(2x - a, 2y, 2z)$, $(2x, 2y - b, 2z)$. In the intersection of these surfaces, $ax = by$; introduce this relation in the inner product of the above vectors to show that this inner product is zero.

13. **a.** Let $\alpha(t)$ be a curve on S with $\alpha(0) = p$, $\alpha'(0) = w$. Then

$$df_p(w) = \frac{d}{dt}(\langle \alpha(t) - p_0, \alpha(t) - p_0 \rangle^{1/2})|_{t=0} = \frac{\langle w, p - p_0 \rangle}{|p - p_0|}.$$

It follows that p is a critical point of f if and only if $\langle w, p - p_0 \rangle = 0$ for all $w \in T_p(S)$.

14. **a.** $f(t)$ is continuous in the interval $(-\infty, c)$, and $\lim_{t\to-\infty} f(t) = 0$, $\lim_{t\to c, t<c} f(t) = +\infty$. Thus, for some $t_1 \in (-\infty, c)$, $f(t_1) = 1$. By similar arguments, we find real roots $t_2 \in (c, b)$, $t_3 \in (b, a)$.

b. The condition for the surfaces $f(t_1) = 1$, $f(t_2) = 1$ to be orthogonal is

$$f_x(t_1)f_x(t_2) + f_y(t_1)f_v(t_2) + f_z(t_1)f_z(t_2) = 0.$$

This reduces to

$$\frac{x^2}{(a-t_1)(a-t_2)} + \frac{y^2}{(b-t_1)(b-t_2)} + \frac{z^2}{(c-t_1)(c-t_2)} = 0,$$

which follows from the fact that $t_1 \neq t_2$ and $f(t_1) - f(t_2) = 0$.

17. Since every surface is locally the graph of a differentiable function, S_1 is given by $f(x, y, z) = 0$ and S_2 by $g(x, y, z) = 0$ in a neighborhood of p; here 0 is a regular value of the differentiable functions f and g. In this neighborhood of p, $S_1 \cap S_2$ is given as the inverse image of (0, 0) of the map $F\colon R^3 \to R^2\colon F(q) = (f(q), g(q))$. Since S_1 and S_2 intersect transversally, the normal vectors (f_x, f_y, f_z) and (g_x, g_y, g_z) are linearly independent. Thus, (0, 0) is a regular value of F and $S_1 \cap S_2$ is a regular curve (cf. Exercise 17, Sec. 2-2).

20. The equation of the tangent plane at (x_0, y_0, z_0) is

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1.$$

The line through O and perpendicular to the tangent plane is given by

$$\frac{xa^2}{x_0} = \frac{yb^2}{y_0} = \frac{zc^2}{z_0}.$$

From the last expression, we obtain

$$\frac{x^2a^2}{xx_0} = \frac{y^2b^2}{yy_0} = \frac{z^2c^2}{zz_0} = \frac{a^2x^2 + b^2y^2 + c^2z^2}{xx_0 + yy_0 + zz_0}.$$

From the same expression, and taking into account the equation of the ellipsoid, we obtain

$$\frac{xx_0}{x_0^2/a^2} = \frac{yy_0}{y_0^2/b^2} = \frac{zz_0}{z_0^2/c^2} = \frac{xx_0 + yy_0 + zz_0}{1}.$$

Again from the same expression and using the equation of the tangent plane, we obtain

$$\frac{x^2}{(x_0x)/a^2} = \frac{y^2}{(y_0y)/b^2} = \frac{z^2}{(z_0z)/c^2} = \frac{x^2 + y^2 + z^2}{1}.$$

The right-hand sides of the three last equations are therefore equal, and hence the asserted equation.

21. Imitate the proof of Prop. 9 of the appendix to Chap. 2.

22. Let r be the fixed line which is met by the normals of S and let $p \in S$. The plane P_1, which contains p and r, contains all the normals to S at the points of $P_1 \cap S$. Consider a plane P_2 passing through p and perpendicular to r. Since the normal through p meets r, P_2 is transversal to $T_p(S)$; hence, $P_2 \cap S$ is a regular plane curve C in a neighborhood of p (cf. Exercise 17, Sec. 2-4). Furthermore $P_1 \cap P_2$ is perpendicular to $T_p(S) \cap P_2$; hence, $P_1 \cap P_2$ is normal to C. It follows that the normals of C all pass through a fixed point $q = r \cap P_2$; hence, C is contained in a circle (cf. Exercise 4, Sec. 1-5). Thus, every $p \in S$ has a neighborhood contained in some surface of revolution with axis r.

SECTION 2-5

8. Since $\partial E/\partial v = 0$, $E = E(u)$ is a function of u alone. Set $\bar{u} = \int \sqrt{E}\, du$. Similarly, $G = G(v)$ is a function of v alone, and we can set $\bar{v} = \int \sqrt{G}\, dv$. Thus, $\bar{u}$ and $\bar{v}$ measure arc lengths along the coordinate curves, whence $\bar{E} = \bar{G} = 1$, $\bar{F} = \cos\theta$.

9. Parametrize the generating curve by arc length.

SECTION 3-2

13. Since the osculating plane is normal to N, $N' = \tau n$ and, therefore, $\tau^2 = |N'|^2 = k_1^2 \cos^2\theta + k_2^2 \sin^2\theta$, where θ is the angle of e_1 with the tangent to the curve. Since the direction is asymptotic, we obtain $\cos^2\theta$ and $\sin^2\theta$ as functions of k_1 and k_2, which substituted in the expression above yields $\tau^2 = -k_1 k_2$.

14. By setting $\lambda_1 = \lambda_1 N_2$ and $\lambda_2 = \lambda_2 N_1$ we have that

$$\begin{aligned}|\lambda_1 - \lambda_2| &= k|\langle n, N_1\rangle N_2 - \langle n, N_2\rangle N_1| \\ &= \sqrt{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\cos\theta}.\end{aligned}$$

On the other hand,

$$\begin{aligned}|\sin\theta| &= |N_1 \wedge N_2| = |n \wedge (N_1 \wedge N_2)| \\ &= |\langle n, N_2\rangle N_1 - \langle n, N_1\rangle N_2|.\end{aligned}$$

16. Intersect the torus by a plane containing its axis and use Exercise 15.

18. Use the fact that if $\theta = 2\pi/m$, then

$$\sigma(\theta) = 1 + \cos^2\theta + \cdots + \cos^2(m-1)\theta = \frac{m}{2},$$

which may be proved by observing that

$$\sigma(\theta) = \frac{1}{4}\left(\sum_{v=-(m-1)}^{v=m-1} e^{2vi\theta} + 2m + 1\right)$$

and that the expression under the summation sign is the sum of a geometric progression, which yields

$$\frac{\sin(2m\theta - \theta)}{\sin\theta} = -1.$$

19. **a.** Express t and h in the basis $\{e_1, e_2\}$ given by the principal directions, and compute $\langle dN(t), h\rangle$.

b. Differentiate $\cos\theta = \langle N, n\rangle$, use that $dN(t) = -k_n t + \tau_g h$, and observe that $\langle N, b\rangle = \langle h, N\rangle = \sin\theta$, where b is the binormal vector.

20. Let S_1, S_2, and S_3 be the surfaces that pass through p. Show that the geodesic torsions of $C_1 = S_2 \cap S_3$ relative to S_2 and S_3 are equal; it will be denoted by τ_1. Similarly, τ_2 denotes the geodesic torsion of $C_2 = S_1 \cap S_3$ and τ_3 that of $S_1 \cap S_2$. Use the definition of τ_g to show that, since C_1, C_2, C_3 are pairwise orthogonal, $\tau_1 + \tau_2 = 0$, $\tau_2 + \tau_3 = 0$, $\tau_3 + \tau_1 = 0$. It follows that $\tau_1 = \tau_2 = \tau_3 = 0$.

SECTION 3-3

2. Asymptotic curves: $u = \text{const.}$, $v = \text{const.}$ Lines of curvature:

$$\log(v + \sqrt{v^2 + c^2}) \pm u = \text{const.}$$

3. $u + v = \text{const.}$ $u - v = \text{const.}$

6. **a.** By taking the line r as the z axis and a normal to r as the x axis, we have that

$$z' = \frac{\sqrt{1 - x^2}}{x}.$$

By setting $x = \sin\theta$, we obtain

$$z(\theta) = \int \frac{\cos^2\theta}{\sin\theta}\, d\theta = \log\tan\frac{\theta}{2} + \cos\theta + C.$$

If $z(\pi/2) = 0$, then $C = 0$.

8. a. The assertion is clearly true if $\mathbf{x} = \mathbf{x}_1$ and $\bar{\mathbf{x}} = \bar{\mathbf{x}}_1$ are parametrizations that satisfy the definition of contact. If $\mathbf{x}$ and $\bar{\mathbf{x}}$ are arbitrary, observe that $\mathbf{x} = \mathbf{x}_1 \circ h$, where h is the change of coordinates. It follows that the partial derivatives of $f \circ \mathbf{x} = f \circ \mathbf{x}_1 \circ h$ are linear combinations of the partial derivatives of $f \circ \mathbf{x}_1$. Therefore, they become zero with the latter ones.

b. Introduce parametrizations $\mathbf{x}(x, y) = (x, y, f(x, y))$ and $\bar{\mathbf{x}}(x, y) = (x, y, \bar{f}(x, y))$, and define a function $h(x, y, z) = f(x, y) - z$. Observe that $h \circ \mathbf{x} = 0$ and $h \circ \bar{\mathbf{x}} = f - \bar{f}$. It follows from part a, applied the function h, that $f - \bar{f}$ has partial derivatives of order ≤ 2 equal to zero at $(0, 0)$.

d. Since contact of order ≥ 2 implies contact of order ≥ 1, the paraboloid passes through p and is tangent to the surface at p. By taking the plane $T_p(S)$ as the xy plane, the equation of the paraboloid becomes

$$\bar{f}(x, y) = ax^2 + 2bxy + cy^2 + dx + ey.$$

Let $z = f(x, y)$ be the representation of the surface in the plane $T_p(S)$. By using part b, we obtain that $d = c = 0$, $a = \frac{1}{2} f_{xx}$, $b = f_{xy}$, $c = \frac{1}{2} f_{yy}$.

15. If there exists such an example, it may locally be written in the form $z = f(x, y)$, with $f(0, 0) = 0$, $f_x(0, 0) = f_y(0, 0) = 0$. The given conditions require that $f_{xx}^2 + f_{yy}^2 \neq 0$ at $(0, 0)$ and that $f_{xx} f_{yy} - f_{xy}^2 = 0$ if and only if $(x, y) = (0, 0)$.

By setting, tentatively, $f(x, y) = \alpha(x) + \beta(y) + xy$, where $\alpha(x)$ is a function of x alone and $\beta(y)$ is a function of y alone, we verify that $\alpha_{xx} = \cos x$, $\beta_{yy} = \cos y$ satisfy the conditions above. It follows that

$$f(x, y) = \cos x + \cos y + xy - 2$$

is such an example.

16. Take a sphere containing the surface and decrease its radius continuously. Study the normal sections at the point (or points) where the sphere meets the surface for the first time.

19. Show that the hyperboloid contains two one-parameter families of lines which are necessarily the asymptotic lines. To find such families of lines, write the equation of the hyperboloid as

$$(x + z)(x - z) = (1 - y)(1 + y)$$

and show that, for each $k \neq 0$, the line $x + z = k(1 + y)$, $x - z = (1/k)(1 - y)$ belongs to the surface.

20. Observe that $(x/a^2, y/b^2, z/c^2) = fN$ for some function f and that an umbilical point satisfies the equation

$$\left\langle \frac{d(fN)}{dt} \wedge \frac{d\alpha}{dt}, N \right\rangle = 0$$

for every curve $\alpha(t) = (x(t), y(t), z(t))$ on the surface. Assume that $z \neq 0$, multiply this equation by z/c^2, and eliminate z and dz/dt (observe that the equation holds for every tangent vector on the surface). Four umbilical points are found, namely,

$$y = 0, \quad x^2 = a^2\frac{a^2 - b^2}{a^2 - c^2}, \quad z^2 = c^2\frac{b^2 - c^2}{a^2 - c^2}.$$

The hypothesis $z = 0$ does not yield any further umbilical points.

21. a. Let $dN(v_1) = av_1 + bv_2$, $dN(v_2) = cv_1 + dv_2$. A direct computation yields

$$\langle d(fN)(v_1) \wedge d(fN)(v_2), fN\rangle = f^3 \det(dN).$$

b. Show that $fN = (x/a^2, y/b^2, z/c^2) = W$, and observe that

$$d(fN)(v_1) = \left(\frac{\alpha_i}{a^2}, \frac{\beta_i}{b^2}, \frac{\gamma_i}{c^2}\right), \quad \text{where } v_1 = (\alpha_i, \beta_i, \gamma_i),$$

$i = 1, 2$. By choosing v_1 so that $v_1 \wedge v_2 = N$, conclude that

$$\langle d(fN)(v_1) \wedge df(N)(v_2), fN\rangle = \frac{\langle W, X\rangle}{a^2b^2c^2}\frac{1}{f},$$

where $X = (x, y, z)$, and therefore $\langle W, X\rangle = 1$.

24. d. Choose a coordinate system in R^3 so that the origin O is at $p \in S$, the xy plane agrees with $T_p(S)$, and the positive direction of the z axis agrees with the orientation of S at p. Furthermore, choose the x and y axes in $T_p(S)$ along the principal directions at p. If V is sufficiently small, it can then be represented as the graph of a differentiable function

$$z = f(x, y), \quad (x, y) \in D \subset R^2,$$

where D is an open disk in R^2 and

$$f_x(0, 0) = f_y(0, 0) = f_{xy}(0, 0) = 0, \quad f_{xx}(0, 0) = k_1, \quad f_{yy}(0, 0) = k_2.$$

We can assume, without loss of generality, that $k_1 \geq 0$ and $k_2 \geq 0$ on D, and we want to prove that $f(x, y) \geq 0$ on D.

Assume that, for some $(\bar{x}, \bar{y}) \in D$, $f(\bar{x}, \bar{y}) < 0$. Consider the function $h_0(t) = f(t\bar{x}, t\bar{y})$, $0 \le t \le 1$. Since $h_0'(0) = 0$, there exists a t_1, $0 \le t_1 \le 1$, such that $h_0''(t_1) < 0$, Let $p_1 = (t_1\bar{x}, t_1\bar{y}, f(t_1\bar{x}, t_1\bar{y})) \in S$, and consider the height function h_1 of V relative to the tangent plane $T_{p_1}(S)$ at p_1. Restricted to the curve $\alpha(t) = (t\bar{x}, t\bar{y}, f(t\bar{x}, t\bar{y}))$, this height function is $h_1(t) = \langle \alpha(t) - p_1, N_1 \rangle$, where N_1 is the unit normal vector at p_1. Thus, $h_1''(t) = \langle \alpha''(t), N_1 \rangle$, and, at $t = t_1$,

$$h_1''(t_1) = \langle (0, 0, h_0''(t_1)), (-f_x(p_1), -f_y(p_1), 1) \rangle = h_0''(t_1) < 0.$$

But $h_1''(t_1) = \langle \alpha''(t_1), N_1 \rangle$ is, up to a positive factor, the normal curvature at p_1, in the direction of $\alpha'(t_1)$. This is a contradiction.

SECTION 3-4

10. c. Reduce the problem to the fact that if λ is an irrational number and m and n run through the integers, the set $\{\lambda m + n\}$ is dense in the real line. To prove the last assertion, it suffices to show that the set $\{\lambda m + n\}$ has arbitrarily small positive elements. Assume the contrary, show that the greatest lower bound of the positive elements of $\{\lambda m + n\}$ still belongs to that set, and obtain a contradiction.

11. Consider the set $\{\alpha_i\colon I_i \to U\}$ of trajectories of w, with $\alpha_i(0) = p$, and set $I = \bigcup_i I_i$. By uniqueness, the maximal trajectory $\alpha\colon I \to U$ may be defined by setting $\alpha(t) = \alpha_i(t)$, where $t \in I_i$.

12. For every $q \in S$, there exist a neighborhood U of q and an interval $(-\epsilon, \epsilon)$, $\epsilon > 0$, such that the trajectory $\alpha(t)$, with $\alpha(0) = q$, is defined in $(-\epsilon, \epsilon)$. By compactness, it is possible to cover S with a finite number of such neighborhoods. Let $\epsilon_0 =$ minimum of the corresponding ϵ's. If $\alpha(t)$ is defined for $t < t_0$ and is not defined for t_0, take $t_1 \in (0, t_0)$, with $|t_0 - t_1| < \epsilon_0/2$. Consider the trajectory $\beta(t)$ of w, with $\beta(t_1) = \alpha(t_1)$, and obtain a contradiction.

SECTION 4-2

3. The "only if" part is immediate.To prove the "if" part, let $p \in S$ and $v \in T_p(S)$, $v \neq 0$. Consider a curve $\alpha\colon (-\epsilon, \epsilon) \to S$, with $\alpha'(0) = v$. We claim that $|d\varphi_p(\alpha'(0))| = |\alpha'(0)|$. Otherwise, say, $|d\varphi_p(\alpha'(0))| > |\alpha'(0)|$, and in a neighborhood J of 0 in $(-\epsilon, \epsilon)$, we have $|d\varphi_{\alpha(t)}(\alpha'(t))| > |\alpha'(t)|$. This implies that the length of $\varphi \circ \alpha(J)$ is greater than the length of $\alpha(J)$, a contradiction.

6. Parametrize α by arc length s in a neighborhood of t_0. Construct in the plane a curve with curvature $k = k(s)$ and apply Exercise 5.

8. Set $0 = (0, 0, 0)$, $G(0) = p_0$, and $G(p) - p_0 = F(p)$. Then $F: R^3 \to R^3$ is a map such that $F(0) = 0$ and $|F(p)| = |G(p) - G(0)| = |p|$. This implies that F preserves the inner product of R^3. Thus, it maps the basis

$$\{(1, 0, 0) = f_1, (0, 1, 0) = f_2, (0, 0, 1) = f_3\}$$

onto an orthonormal basis, and if $p = \sum a_i f_i$, $i = 1, 2, 3$, then $F(p) = \sum \alpha_i F(f_i)$. Therefore, F is linear.

11. **a.** Since F is distance-preserving and the arc length of a differentiable curve is the limit of the lengths of inscribed polygons, the restriction $F|S$ preserves the arc length of a curve in S.

c. Consider the isometry of an open strip of the plane onto a cylinder minus a generator.

12. The restriction of $F(x, y, z) = (x, -y, -z)$ to C is an isometry of C (cf. Exercise 11), the fixed points of which are $(1, 0, 0)$ and $(-1, 0, 0)$.

17. The loxodromes make a constant angle with the meridians of the sphere. Under Mercator's projection (see Exercise 16) the meridians go into parallel straight lines in the plane. Since Mercator's projection is conformal, the loxodromes also go into straight lines. Thus, the sum of the interior angles of the triangle in the sphere is the same as the sum of the interior angles of a rectilinear plane triangle.

SECTION 4-4

6. Use the fact that the absolute value of the geodesic curvature is the absolute value of the projection onto the tangent plane of the usual curvature.

8. Use Exercise 1, part b, and Prop. 4 of Sec. 3-2.

9. Use the fact that the meridians are geodesics and that the parallel transport preserves angles.

10. Apply the relation $k_g^2 + k_n^2 = k^2$ and the Meusnier theorem to the projecting cylinder.

12. Parametrize a neighborhood of $p \in S$ in such a way that the two families of geodesics are coordinate curves (Corollary 1, Sec. 3-4). Show that this implies that $F = 0$, $E_v = 0 = G_u$. Make a change of parameters to obtain that $\bar{F} = 0$, $\bar{E} = \bar{G} = 1$.

13. Fix two orthogonal unit vectors $v(p)$ and $w(p)$ in $T_p(S)$ and parallel transport them to each point of V. Two differentiable, orthogonal, unit vector fields are thus obtained. Parametrize V in such a way that the directions of these vectors are tangent to the coordinate curves, which are then geodesics. Apply Exercise 12.

16. Parametrize a neighborhood of $p \in S$ in such a way that the lines of curvature are the coordinate curves and that $v = \text{const.}$ are the asymptotic curves. It follows that $e_v = 0$, and from the Mainardi-Codazzi equations, we conclude that $E_v = 0$. This implies that the geodesic curvature of $v = \text{const.}$ is zero. For the example, look at the upper parallel or the torus.

18. Use Clairaut's relation (cf. Example 5).

19. Substitute in Eq. (4) the Christoffel symbols by their values as functions of E, F, and G and differentiate the expression of the first fundamental form:

$$1 = E(u')^2 + 2Fu'v' + G(v')^2.$$

20. Use Clairaut's relation.

SECTION 4-5

4. b. Observe that the map $x = \bar{x}$, $y = (\bar{y})^5$, $z = (\bar{z})^3$ gives a homeomorphism of the sphere $x^2 + y^2 + z^2 = 1$ onto the surface $(\bar{x})^2 + (\bar{y})^{10} + (\bar{z})^6 = 1$.

6. a. Restrict v to the curve $\alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. The angle that $v(t)$ forms with the x axis is t. Thus, $2\pi I = 2\pi$; hence, $I = 1$.

d. By restricting v to the curve $\alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, we obtain $v(t) = (\cos^2 t - \sin^2 t, -2\cos t \sin t) = (\cos 2t, -\sin 2t)$. Thus, $I = -2$.

SECTION 4-6

8. Let (ρ, θ) be a system of geodesic polar coordinates such that its pole is one of the vertices of Δ and one of the sides of Δ corresponds to $\theta = 0$. Let the two other sides be given by $\theta = \theta_0$ and $\rho = h(\theta)$. Since the vertex that corresponds to the pole does not belong to the coordinate neighborhood, take a small circle of radius ϵ around the pole. Then

$$\iint_\Delta K\sqrt{G}\, d\rho\, d\theta = \int_0^{\theta_0} d\theta \left(\lim_{\epsilon \to 0} \int_\epsilon^{h(\theta)} K\sqrt{G}\, d\rho \right).$$

Observing that $K\sqrt{G} = -(\sqrt{G})_{\rho\rho}$ and that $\lim_{\epsilon\to 0}(\sqrt{G})_\rho = 1$, we have that the limit enclosed in parentheses is given by

$$1 - \frac{\partial(\sqrt{G})}{\partial\rho}(h(\theta), \theta).$$

By using Exercise 7, we obtain

$$\iint_\Delta K\sqrt{G}\, d\rho\, d\theta = \int_0^{\theta_0} d\theta - \int_0^{\theta_0} d\varphi$$
$$= \alpha_3 - (\pi - \alpha_2 - \alpha_1) = \sum_1^3 \alpha_i - \pi.$$

12. c. For $K \equiv 0$, the problem is trivial. For $K > 0$, use part b. For $K < 0$, consider a coordinate neighborhood V of the pseudosphere (cf. Exercise 6, part b, Sec. 3-3), parametrized by polar coordinates (ρ, θ); that is, $E = 1$, $F = 0$, $G = \sinh^2 \rho$. Compute the geodesics of V; it is convenient to use the change of coordinates $\tanh\rho = 1/w$, $\rho \neq 0$, $\theta = \theta$, so that

$$E = \frac{1}{(w^2-1)^2}, \quad G = \frac{1}{w^2-1}, \quad F = 0,$$
$$\Gamma^1_{11} = -\frac{2w}{w^2-1}, \quad \Gamma^1_{12} = -\frac{w}{w^2-1}, \quad \Gamma^1_{22} = w,$$

and the other Christoffel symbols are zero. It follows that the nonradial geodesics satisfy the equation $(d^2w/d\theta^2) + w = 0$, where $w = w(\theta)$. Thus, $w = A\cos\theta + B\sin\theta$; that is

$$A\tanh\rho\cos\theta + B\tanh\rho\sin\theta = 1.$$

Therefore, the map of V into R^2 given by

$$\xi = \tanh\rho\cos\theta, \quad \eta = \tanh\rho\sin\theta,$$

$(\xi, \eta) \in R^2$, is a geodesic mapping.

13. b. Define $\mathbf{x} = \varphi^{-1}: \varphi(U) \subset R^2 \to S$. Let $v = v(u)$ be a geodesic in U. Since φ is a geodesic mapping and the geodesics of R^2 are lines, then $d^2v/du^2 \equiv 0$. By bringing this condition into part a, the required result is obtained.

c. Equation (a) is obtained from Eq. (5) of Sec. 4-3 using part b. From Eq. (5a) of Sec. 4-3 together with part b we have

$$KF = (\Gamma^1_{12})_u - 2(\Gamma^2_{12})_v + \Gamma^2_{12}\Gamma^1_{12}.$$

By interchanging u and v in the expression above and subtracting the results, we obtain $(\Gamma^1_{12})_u = (\Gamma^2_{12})_v$, whence Eq. (b). Finally, Eqs. (c) and (d) are obtained from Eqs. (a) and (b), respectively, by interchanging u and v.

d. By differentiating Eq. (a) with respect to v, Eq. (b) with respect to u, and subtracting the results, we obtain

$$EK_v - FK_u = -K(E_v - F_u) + K(-F\Gamma_{12}^2 + E\Gamma_{12}^1).$$

By taking into account the values of Γ_{ij}^k, the expression above yields

$$EK_v - FK_u = -K(E_v - F_u) + K(E_v - F_u) = 0.$$

Similarly, from Eqs. (c) and (d) we obtain $FK_v - GK_u = 0$, whence $K_v = K_u = 0$.

SECTION 4-7

1. Consider an orthonormal basis $\{e_1, e_2\}$ at $T_{\alpha(0)}(S)$ and take the parallel transport of e_1 and e_2 along α, obtaining an orthonormal basis $\{e_1(t), e_2(t)\}$ at each $T_{\alpha(t)}(S)$. Set $w(\alpha(t)) = w_1(t)e_1(t) + w_2(t)e_2(t)$. Then $D_y w = w_1'(0)e_1 + w_2'(0)e_2$ and the second member is the velocity of the curve $w_1(t)e_1 + w_2(t)e_2$ in $T_p(S)$ at $t = 0$.

2. b. Show that if $(t_1, t_2) \subset I$ is small and does not contain "break points of α," then the tangent vector field of $\alpha((t_1, t_2))$ can be extended to a vector field y in a neighborhood of $\alpha((t_1, t_2))$. Thus, by restricting v and w to α, property 3 becomes

$$\frac{d}{dt}\langle v(t), w(t)\rangle = \left\langle \frac{Dv}{dt}, w\right\rangle + \left\langle v, \frac{Dw}{dt}\right\rangle,$$

which implies that parallel transport in $\alpha|(t_1, t_2)$ is an isometry. By compactness, this can be extended to the entire I. Conversely, assume that parallel transport is an isometry. Let α be the trajectory of y through a point $p \in S$. Restrict v and w to α. Choose orthonormal basis $\{e_1(t), e_2(t)\}$ as in the solution of Exercise 1, and set $v(t) = v_1e_1 + v_2e_2$, $w(t) = w_1e_1 + w_2e_2$. Then property 3 becomes the "product rule":

$$\frac{d}{dt}\left(\sum_i v_i w_i\right) = \sum_i \frac{dv_i}{dt} w_i + \sum_i v_i \frac{dw_i}{dt}, \quad i = 1, 2.$$

c. Let D be given and choose an orthogonal parametrization $\mathbf{x}(u, v)$. Let $y = y_1\mathbf{x}_u + y_2\mathbf{x}_v$, $w = w_1\mathbf{x}_u + w_2\mathbf{x}_v$. From properties 1, 2, and 3, it follows that $D_y w$ is determined by the knowledge of $D_{\mathbf{x}_u}\mathbf{x}_u$,

$D_{\mathbf{x}_u}\mathbf{x}_v$, $D_{\mathbf{x}_v}\mathbf{x}_v$. Set $D_{\mathbf{x}_u}\mathbf{x}_u = A_{11}^1\mathbf{x}_u + A_{11}^2\mathbf{x}_v$, $D_{\mathbf{x}_u}\mathbf{x}_v = A_{12}^1\mathbf{x}_u + A_{12}^2\mathbf{x}_v$, $D_{\mathbf{x}_v}\mathbf{x}_v = A_{22}^1\mathbf{x}_u + A_{22}^2\mathbf{x}_v$. From property 3 it follows that the A_{ij}^k satisfy the same equations as the Γ_{ij}^k (cf. Eq. (2), Sec. 4-3). Thus, $A_{ij}^k = \Gamma_{ij}^k$, which proves that $D_y v$ agrees with the operation "Take the usual derivative and project it onto the tangent plane."

3. a. Observe that

$$d\mathbf{x}_{(0,t)}(1,0) = \left(\frac{\partial \mathbf{x}}{\partial s}\right)_{s=0} = \frac{d}{ds}\gamma(s, \alpha(t), v(t))\bigg|_{s=0} = v(t),$$
$$d\mathbf{x}_{(0,t)}(0,1) = \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{s=0} = \alpha'(t).$$

b. Use the fact that $\mathbf{x}$ is a local diffeomorphism to cover the compact set I with a family of open intervals in which $\mathbf{x}$ is one-to-one. Use the Heine-Borel theorem and the Lebesgue number of the covering (cf. Sec. 2-7) to globalize the result.

c. To show that $F = 0$, we compute (cf. property 4 of Exercise 2)

$$\frac{d}{ds}F = \frac{d}{ds}\left\langle \frac{\partial \mathbf{x}}{\partial s}, \frac{\partial \mathbf{x}}{\partial t}\right\rangle = \left\langle \frac{D}{\partial s}\frac{\partial \mathbf{x}}{\partial s}, \frac{\partial \mathbf{x}}{\partial t}\right\rangle + \left\langle \frac{\partial \mathbf{x}}{\partial s}, \frac{D}{\partial s}\frac{\partial \mathbf{x}}{\partial t}\right\rangle = \left\langle \frac{\partial \mathbf{x}}{\partial s}, \frac{D}{\partial t}\frac{\partial \mathbf{x}}{\partial s}\right\rangle,$$

because the vector field $\partial\mathbf{x}/\partial s$ is parallel along $t = \text{const}$. Since

$$0 = \frac{d}{dt}\left\langle \frac{\partial \mathbf{x}}{\partial s}, \frac{\partial \mathbf{x}}{\partial s}\right\rangle = 2\left\langle \frac{D}{\partial t}\frac{\partial \mathbf{x}}{\partial s}, \frac{\partial \mathbf{x}}{\partial s}\right\rangle,$$

F does not depend on s. Since $F(0,t) = 0$, we have $F = 0$.

d. This is a consequence of the fact that $F = 0$.

4. a. Use Schwarz's inequality,

$$\left(\int_a^b fg\,dt\right)^2 \le \int_a^b f^2\,dt \int_a^b g^2\,dt,$$

with $f \equiv 1$ and $g = |d\alpha/dt|$.

5. a. By noticing that $E(t) = \int_0^l \{(\partial u/\partial v)^2 + G(\gamma(v,t), v)\}\,dv$, we obtain (we write $\gamma(v,t) = u(v,t)$, for convenience)

$$E'(t) = \int_0^l \left\{2\frac{\partial u}{\partial v}\frac{\partial^2 u}{\partial v \partial t} + \frac{\partial G}{\partial u}u'\right\}dv.$$

Since, for $t = 0$, $\partial u/\partial v = 0$ and $\partial G/\partial u = 0$, we have proved the first part.

Furthermore,

$$E''(t) = \int_0^l \left\{ 2\left(\frac{\partial^2 u}{\partial v \partial t}\right)^2 + 2\frac{\partial u}{\partial v}\frac{\partial^3 u}{\partial v \partial^2 t} + \frac{\partial^2 G}{\partial u^2}(u')^2 + \frac{\partial G}{\partial u}u'' \right\} dv.$$

Hence, by using $G_{uu} = -2K\sqrt{G}$ and noting that $\sqrt{G} = 1$ for $t = 0$, we obtain

$$E''(0) = 2\int_0^l \left\{ \left(\frac{d\eta}{dv}\right)^2 - K\eta^2 \right\} dv.$$

6. b. Choose $\epsilon > 0$ and coordinates in $R^3 \supset S$ so that $\varphi(\rho, \epsilon) = q$. Consider the points $(\rho, \epsilon) = r_0$, $(\rho, \epsilon + 2\pi \sin\beta) = r_1, \dots, (\rho, \epsilon + 2\pi k \sin\beta) = r_k$. Taking ϵ sufficiently small, we see that the line segments $\overline{r_0 r_1}, \dots, \overline{r_0 r_k}$ belong to V if $2\pi k \sin\beta < \pi$ (Fig. 4-49). Since φ is a local isometry, the images of these segments will be geodesics joining q to q, which are clearly broken at q (Fig. 4-49).

c. It must be proved that each geodesic $\gamma\colon [0, l] \to S$ with $\gamma(0) = \gamma(l) = q$ is the image by φ of one of the line segments $\overline{r_0 r_1}, \dots, \overline{r_0 r_k}$ referred to in part b. For some neighborhood $U \subset V$ of r_0, the restriction $\varphi|U = \tilde{\varphi}$ is an isometry. Thus, $\tilde{\varphi}^{-1} \circ \gamma$ is a segment of a half-line L starting at r_0. Since $\varphi(L)$ is a geodesic which agrees with $\gamma([0, l])$ in an open interval, it agrees with γ where γ is defined. Since $\gamma(l) = q$, L passes through one of the points r_i, $i = 1, \dots, k$, say r_j, and so γ is the image of $\overline{r_0 r_j}$.

SECTION 5-2

3. a. Use the relation $\varphi'' = -K\varphi$ to obtain $(\varphi'^2 + K\varphi^2)' = K'\varphi^2$. Integrate both sides of the last relation and use the boundary conditions of the statement.

SECTION 5-3

5. Assume that every Cauchy sequence in d converges and let $\gamma(s)$ be a geodesic parametrized by arc length. Suppose, by contradiction, that $\gamma(s)$ is defined for $s < s_0$ but not for $s = s_0$. Choose a sequence $\{s_n\} \to s_0$. Thus, given $\epsilon > 0$, there exists n_0 such that if $n, m > n_0$, $|s_n - s_m| < \epsilon$. Therefore,

$$d(\gamma(s_m), \gamma(s_n)) \le |s_n - s_m| < \epsilon$$

and $\{\gamma(s_n)\}$ is a Cauchy sequence in d. Let $\{\gamma(s_n)\} \to p_0 \in S$ and let W be a neighborhood of p_0 as given by Prop. 1 of Sec. 4-7. If m, n

are sufficiently large, the small geodesic joining $\gamma(s_m)$ to $\gamma(s_n)$ clearly agrees with γ. Thus, γ can be extended through p_0, a contradiction.

Conversely, assume that S is complete and let $\{p_n\}$ be a Cauchy sequence in d of points on S. Since d is greater than or equal to the Euclidean distance $\bar{d}$, $\{p_n\}$ is a Cauchy sequence in $\bar{d}$. Thus, $\{p_n\}$ converges to $p_0 \in R^3$. Assume, by contradiction, that $p_0 \notin S$. Since a Cauchy sequence is bounded, given $\epsilon > 0$ there exists an index n_0 such that, for all $n > n_0$, the distance $d(p_{n_0}, p_n) < \epsilon$. By the Hopf-Rinow theorem, there is a minimal geodesic γ_n joining p_{n_0} to p_n with length $< \epsilon$. As $n \to \infty$, γ_n tends to a minimal geodesic γ with length $\leq \epsilon$. Parametrize γ by arc length s. Then, since $p_0 \notin S$, γ is not defined for $s = \epsilon$. This contradicts the completeness of S.

6. Let $\{p_n\}$ be a sequence of points on S such that $d(p, p_n) \to \infty$. Since S is complete, there is a minimal geodesic $\gamma_n(s)$ (parametrized by arc length) joining p to p_n with $\gamma_n(0) = p$. The unit vectors $\gamma_n'(0)$ have a limit point v on the (compact) unit sphere of $T_p(S)$. Let $\gamma(s) = \exp_p sv$, $s \geq 0$. Then $\gamma(s)$ is a ray issuing from p. To see this, notice that, for a fixed s_0 and n sufficiently large, $\lim_{n\to\infty} \gamma_n(s_0) = \gamma(s_0)$. This follows from the continuous dependence of geodesics from the initial conditions. Furthermore, since d is continuous,

$$\lim_{n\to\infty} d(p, \gamma_n(s_0)) = d(p, \gamma(s_0)).$$

But if n is large enough, $d(p, \gamma_n(s_0)) = s_0$. Thus, $d(p, \gamma(s_0)) = s_0$, and γ is a ray.

8. First show that if d and $\bar{d}$ denote the intrinsic distances of S and $\bar{S}$, respectively, then $d(p, q) \geq c\bar{d}(\varphi(p), \varphi(q))$ for all $p, q \in S$. Now let $\{p_n\}$ be a Cauchy sequence in d of points on S. By the initial remark, $\{\varphi(p_n)\}$ is a Cauchy sequence in $\bar{d}$. Since $\bar{S}$ is complete, $\{\varphi(p_n)\} \to \varphi(p_0)$. Since φ^{-1} is continuous, $\{p_n\} \to p_0$. Thus, every Cauchy sequence in d converges; hence S is complete (cf. Exercise 5).

9. *φ is one-to-one*: Assume, by contradiction, that $p_1 \neq p_2 \in S_1$ are such that $\varphi(p_1) = \varphi(p_2) = q$. Since S_1 is complete, there is a minimal geodesic γ joining p_1 to p_2. Since φ is a local isometry, $\varphi \circ \gamma$ is a geodesic joining q to itself with the same length as γ. Any point distinct from q on $\varphi \circ \gamma$ can be joined to q by two geodesics, a contradiction.

φ is onto: Since φ is a local diffeomorphism, $\varphi(S_1) \subset S_2$ is an open set in S_2. We shall prove that $\varphi(S_1)$ is also closed in S_2; since S_2 is connected, this will imply that $\varphi(S_1) = S_2$. If $\varphi(S_1)$ is not closed in S_2, there exists a sequence $\{\varphi(p_n)\}$, $p_n \in S_1$, such that $\{\varphi(p_n)\} \to p_0 \in \varphi(S_1)$. Thus, $\{\varphi(p_n)\}$ is a nonconverging Cauchy sequence in $\varphi(S_1)$. Since φ is a one-to-one local isometry, $\{p_n\}$ is a nonconverging Cauchy sequence in S_1, a contradiction to the completeness of S_1.

10. a. Since

$$\frac{d}{dt}(h \circ \varphi(t)) = \frac{d}{dt}\langle \varphi(t), v\rangle = \langle \varphi'(t), v\rangle = \langle \operatorname{grad} h, v\rangle$$

and

$$\frac{d}{dt}(h \circ \varphi(t)) = dh(\varphi'(t)) = dh(\operatorname{grad} h) = \langle \operatorname{grad} h, \operatorname{grad} h\rangle,$$

we conclude, by equating the last members of the above relations, that $|\operatorname{grad} h| \leq 1$.

b. Assume that $\varphi(t)$ is defined for $t < t_0$ but not for $t = t_0$. Then there exists a sequence $\{t_n\} \to t_0$ such that the sequence $\{\varphi(t_n)\}$ does not converge. If m and n are sufficiently large, we use part a to obtain

$$d(\varphi(t_m), \varphi(t_m)) \leq \int_{t_n}^{t_m} |\operatorname{grad} h(\varphi(t))|\, dt \leq |t_m - t_n|,$$

where d is the intrinsic distance of S. This implies that $\{\varphi(t_n)\}$ is a nonconverging Cauchy sequence in d, a contradiction to the completeness of S.

SECTION 5-4

2. Assume that

$$\lim_{r\to\infty}\Big(\inf_{x^2+y^2\geq r} K(x, y)\Big) = 2c > 0.$$

Then there exists $R > 0$ such that if $(x, y) \notin D$, where

$$D = \{(x, y) \in R^2; x^2 + y^2 < R^2\},$$

then $K(x, y) \geq c$. Thus, by taking points outside the disk D, we can obtain arbitrarily large disks where $K(x, y) \geq c > 0$. This is easily seen to contradict Bonnet's theorem.

SECTION 5-5

3. b. Assume that $a > b$ and set $s = b$ in relation $(*)$. Use the initial conditions and the facts $v'(b) < 0$, $u(b) > 0$, $uv \geq 0$ in $[0, b]$ to obtain a contradiction.

c. From $[uv' - vu']_0^s \geq 0$, one obtains $v'/v \geq u'/u$; that is, $(\log v)' \geq (\log u)'$. Now, let $0 < s_0 \leq s \leq a$, and integrate the last inequality between s_0 and s to obtain

$$\log v(s) - \log v(s_0) \geq \log u(s) - \log u(s_0);$$

that is, $v(s)/u(s) \geq v(s_0)/u(s_0)$. Next, observe that

$$\lim_{s_0 \to 0} \frac{v(s_0)}{u(s_0)} = \lim_{s_0 \to 0} \frac{v'(s_0)}{u'(s_0)} = 1.$$

Thus, $v(s) \geq u(s)$ for all $s \in [0, a)$.

6. Suppose, by contradiction, that $u(s) \neq 0$ for all $s \in (0, s_0]$. By using Eq. $(*)$ of Exercise 3, part b (with $\tilde{K} = L$ and $s = s_0$), we obtain

$$\int_0^{s_0} (K - L)uv\, ds + u(s_0)v'(s_0) - u(0)v'(0) = 0.$$

Assume, for instance, that $u(s) > 0$ and $v(s) < 0$ on $(0, s_0]$. Then $v'(0) < 0$ and $v'(s_0) > 0$. Thus, the first term of the above sum is ≥ 0 and the two remaining terms are > 0, a contradiction. All the other cases can be treated similarly.

8. Let $\mathfrak{v}$ be the vector space of Jacobi fields J along γ with the property that $J(l) = 0$. $\mathfrak{v}$ is a two-dimensional vector space. Since $\gamma(l)$ is not conjugate to $\gamma(0)$, the linear map $\theta\colon \mathfrak{v} \to T_{\gamma(0)}(S)$ given by $\theta(J) = J(0)$ is injective, and hence, for dimensional reasons, an isomorphism. Thus, there exists $J \in \mathfrak{v}$ with $J(0) = w_0$. By the same token, there exists a Jacobi field $\bar{J}$ along γ with $\bar{J}(0) = 0$, $\bar{J}(l) = w_1$. The required Jacobi field is given by $J + \bar{J}$.

SECTION 5-6

10. Let $\gamma\colon [0, l] \to S$ be a simple closed geodesic on S and let $v(0) \in T_{\gamma(0)}(S)$ be such that $|v(0)| = 1$, $\langle v(0), \gamma'(0)\rangle = 0$. Take the parallel transport $v(s)$ of $v(0)$ along γ. Since S is orientable, $v(l) = v(0)$ and v defines a differentiable vector field along γ. Notice that v is orthogonal to γ and that $Dv/ds = 0$, $s \in [0, 1)$. Define a variation (with free end points) $h\colon [0, l) \times (-\epsilon, \epsilon) \to S$ by

$$h(s, t) = \exp_{\gamma(s)} tv(s).$$

Check that, for t small, the curves of the variation $h_t(s) = h(s, t)$ are closed. Extend the formula for the second variation of arc length to the present case, and show that

$$L_v''(0) = -\int_0^t K ds < 0.$$

Thus, $\gamma(s)$ is longer than all curves $h_t(s)$ for t small, say, $|t| < \delta \leq \epsilon$. By changing the parameter t into t/δ, we obtain the required homotopy.

SECTION 5-7

9. Use the notion of geodesic torsion τ_g of a curve on a surface (cf. Exercise 19, Sec. 3-2). Since

$$\frac{d\theta}{ds} = \tau - \tau_g,$$

where $\cos\theta = \langle N, n\rangle$ and the curve is closed and smooth, we obtain

$$\int_0^l \tau\, ds - \int_0^l \tau_g\, ds = 2\pi n,$$

where n is an integer. But on the sphere, all curves are lines of curvature. Since the lines of curvature are characterized by having vanishing geodesic torsion (cf. Exercise 19, Sec. 3-2), we have

$$\int_0^l \tau\, ds = 2\pi n.$$

Since every closed curve on a sphere is homotopic to zero, the integer n is easily seen to be zero.

SECTION 5-10

7. We have only to show that the geodesics $\gamma(s)$ parametrized by arc length which approach the boundary of R^2_+ are defined for all values of the parameter s. If the contrary were true, such a geodesic would have a finite length l, say, from a fixed point p_0. But for the circles of R^2_+ that are geodesics, we have

$$l = \left|\lim_{\epsilon\to 0}\int_{\theta_0>\pi/2}^{\epsilon} \frac{d\theta}{\sin\theta}\right| \geq \left|\lim_{\epsilon\to 0}\int_{\theta_0>\pi/2}^{\epsilon} \frac{\cos\theta\, d\theta}{\sin\theta}\right| = \infty,$$

and the same holds for the vertical lines of R^2_+.

10. **c.** To prove that the metric is complete, notice first that it dominates the Euclidean metric on R^2. Thus, if a sequence is a Cauchy sequence in the given metric, it is also a Cauchy sequence in the Euclidean metric. Since the Euclidean metric is complete, such a sequence converges. It follows that the given metric is complete (cf. Exercise 1, Sec. 5-3).